Asymptotic Controllability Implies Feedback Stabilization

F.H. Clarke
Yu.S. Ledyaev
E.D. Sontag
A.I. Subbotin

Abstract—
It is shown that every asymptotically controllable system can be globally stabilized by means of some (discontinuous) feedback law. The stabilizing strategy is based on pointwise optimization of a smoothed version of a control-Lyapunov function, iteratively sending trajectories into smaller and smaller neighborgoods of a desired equilibrium. A major technical problem, and one of contributions of the present paper, concerns the precise meaning of "solution" when using a discontinuous controller.

I. Introduction

A longstanding open question in nonlinear control theory concerns the relationship between asymptotic controllability to the origin in $\mathbb{R}^n$ of a nonlinear system

$$\dot{x} \;=\; f(x,u) \tag{1}$$

by an "open loop" control $u : [0,+\infty) \to \mathbb{U}$ and the existence of a feedback control $k : \mathbb{R}^n \to \mathbb{U}$ which stabilizes trajectories of the system

$$\dot{x} \;=\; f(x,k(x)) \tag{2}$$

with respect to the origin.

For the special case of linear control systems $\dot{x} = Ax + Bu$, this relationship is well understood: asymptotic controllability is equivalent to the existence of a continuous (even linear) stabilizing feedback law. But it is well-known that continuous feedback laws may fail to exist even for simple asymptotically controllable nonlinear systems. This is especially easy to see, as discussed in [27], for one-dimensional ($\mathbb{U}$=$\mathbb{R}$, n=1) systems (1): in that case asymptotic controllability is equivalent to the property "for each $x \neq 0$ there is some value u so that $xf(x,u) < 0$", but it is easy to construct examples of functions f, even analytic, for which this property is satisfied but for which no possible continuous section $k : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ exists so that $xf(x,k(x)) < 0$ for all nonzero x. General results regarding the nonexistence of continuous feedback were presented in the paper [3], where techniques from topological degree theory were used (an exposition is given in the textbook [25]).

These negative results led to the search for feedback laws which are not necessarily of the form $u = k(x)$, k a continuous function. One possible approach consists of looking for dynamical feedback laws, where additional "memory" variables are introduced into a controller, and as a very special case, time-varying (even periodic) continuous feedback $u = k(t,x)$. Such time-varying laws were shown in [27] to be always possible in the case of one-dimensional systems, and in the major work [9] (see also [10]) it was shown that they are also always possible when the original system is completely controllable and has "no drift", meaning essentially that $f(x,0) = 0$ for all states (see also [26] for numerical algorithms and an alternative proof of the time-varying result for analytic systems). However, for the general case of asymptotically controllable systems with drift, no dynamic or time-varying solutions are known. Thus it is natural to ask about the existence of *discontinuous* feedback laws $u = k(x)$. Such feedbacks are often obtained when solving optimal-control problems, for example, so it is interesting to search for general theorems insuring their existence. Unfortunately, allowing nonregular feedback leads to an immediate difficulty: how should one define the meaning of *solution* $x(\cdot)$ of the differential equation (2) with discontinuous right-hand side?

One of the best-known candidates for the concept of solution of (2) is that of a *Filippov solution* (cf. [13]), which is defined as the solution of a certain differential inclusion with multivalued right-hand side which is built from $f(x,k(x))$. However, it follows from the results in [21], [11] that the existence of a discontinuous stabilizing feedback in the Filippov sense implies the same Brockett necessary conditions [3] as the existence of a continuous stabilizing feedback does. Moreover, it is shown in [11] that the existence of a stabilizing feedback in the Filippov sense is equivalent to the existence of a continuous stabilizing one, in the case of systems affine in controls. In conclusion, there is no hope of obtaining general results if one insists

F.H. Clarke is with Universite' de Lyon I (BAT 101), 69622 Villeurbanne, France, and CRM, Université de Montréal, Montréal (Qc) Canada H3C 3J7; email: clarke@crm.umontreal.ca. His work was supported in part by the Natural Sciences and Engineering Research Council of Canada and Le Fonds FCAR du Québec

Yu.S. Ledyaev was visiting Rutgers University when this work was done. His permanent address is Steklov Institute of Mathematics, Moscow 117966, Russia; email: ledyaev@mian.su. His work was supported in part by Russian Fund for Fundamental Research Grant 96-01-00219, by NSERC and La Fonde FCAR du Québec, and by the Rutgers Center for Systems and Control (SYCON)

E.D. Sontag is with Dept. of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA; email: sontag@control.rutgers.edu. His work was supported in part by US Air Force Grant AFOSR-94-0293

A.I. Subbotin is with Inst. of Mathematics & Mechanics, Ekaterinburg 620219, Russia; email: sai@subb.intec.ru. His work was supported in part by Russian Fund for Fundamental Research Grant 96-01-00219

on the use of Filippov solutions.

In this paper, we develop a concept of solution of (2) for arbitrary feedback $k(x)$ which has (a) a clear and reasonable physical meaning (perhaps even more so than the definitions derived from differential inclusions), and (b) allows proving the desired general theorem. Our notion is borrowed from the theory of positional differential games, and it was systematically studied in that context by Krasovskii and Subbotin in [19].

There have been several papers dealing with rather general theorems on discontinuous stabilization. One of the best known is [30], which provided piecewise analytic feedback laws for analytic systems which satisfy controllability conditions. The definition of "feedback" given in that paper involves a specification of "exit rules" for certain lower-dimensional submanifolds, and these cannot be expressed in terms of a true feedback law (even in the sense of this paper). *Sampling* is a strategy commonly used in digital control (see e.g. [25] for a discussion of sampling in a general nonlinear systems context), and an approach to sampled control of nonlinear systems was given in [22]. Sampling is not true feedback, in that one typically uses a fixed sampling rate, or perhaps a predetermined sampling schedule, and intersample behavior is not accounted for. A stabilization approach introduced in [16] is based on a sampling-like method under strong controllability Lie algebraic conditions on the system being controlled. One may interpret our solutions as representing the behavior of sampling, with a fixed feedback law being used, as the sampling periods tend to zero – indeed, such a speedup of sampling is essential as we approach the target state, to avoid an overshoot during the sampling interval, as well as far from the target, due to possible explosion times in the dynamics.

A. Definition of Feedback Solution

From now on, we assume that $\mathbb{U}$ is a locally compact metric space and that the mapping $f : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n : (x,u) \mapsto f(x,u)$ is continuous, and locally Lipschitz on x uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{U}$. We use $|x|$ to denote the usual Euclidean norm of $x \in \mathbb{R}^n$, and $\langle x, y\rangle$ for the inner product of two such vectors.

A locally bounded (for each compact, image is relatively compact) function $k : \mathbb{R}^n \to \mathbb{U}$ will be called a *feedback*.

Any infinite sequence $\pi = \{t_i\}_{i\geq 0}$ consisting of numbers

$$0 = t_0 < t_1 < t_2 < \ldots$$

with $\lim_{i\to\infty} t_i = \infty$ is called a *partition* (of $[0, +\infty)$), and the number

$$d(\pi) := \sup_{i\geq 0}(t_{i+1} - t_i)$$

is its *diameter*.

We next define the trajectory associated to a feedback $k(x)$ and any given partition π as the solution obtained by means of the following procedure: on each interval $[t_i, t_{i+1}]$, the initial state is measured, $u_i = k(x(t_i))$ is computed, and then the constant control $u \equiv u_i$ is applied until time t_{i+1}, when a new measurement is taken. This notion of solution is an accurate model of the process used in computer control ("sampling").

Definition I.1: Assume given a feedback k, a partition π, and an $x_0 \in \mathbb{R}^n$. For each i, $i = 0, 1, 2, \ldots$, recursively solve

$$\dot{x}(t) \;=\; f(x(t), k(x(t_i)))\,,\; t \in [t_i, t_{i+1}] \qquad (3)$$

using as initial value $x(t_i)$ the endpoint of the solution on the preceding interval (and starting with $x(t_0) = x_0$). The π-*trajectory* of (2) *starting from* x_0 is the function $x(\cdot)$ thus obtained.

Observe that this solution may fail to be defined on all of $[0, +\infty)$, because of possible finite escape times in one of the intervals, in which case we only have a trajectory defined on some maximal interval. In our results, however the construction will provide a feedback for which solutions are globally defined; we say in that case that the trajectory *is well-defined*.

Remark I.2: It is worth pointing out that the concept of solution introduced above is quite different from the "Euler solution" which would be obtained when attempting to solve the differential equation (2) using Euler's method: in that case, on each interval $[t_i, t_{i+1}]$ one would have the formula $x(t) = x(t_i) + (t - t_i)f(x(t_i), k(x(t_i)))$, corresponding to the solution of the different equation $\dot{x} = f(x(t_i), k(x(t_i)))$. *This alternative definition does not have any physical meaning in terms of the original system.* (It is amusing to note, however, that this could be interpreted as the π-trajectory of the system $\dot{x} = u$ under the feedback $u = f(x, k(x))$, so in that sense, Euler approximations are a particular case of our setup.) □

B. Statement of Main Result

The main objective of this paper is to explore the relationship between the existence of stabilizing (discontinuous) feedback and asymptotic controllability of the open loop system (1). We first define the meaning of (globally) stabilizing feedback. This is a feedback law which, for fast enough sampling, drives all states asymptotically to the origin and with small overshoot. Of course, since sampling is involved, when near the origin it is impossible to guarantee arbitrarily small displacements unless a faster sampling rate is used, and, for technical reasons (for instance, due to the existence of possible explosion times), one might also need to sample faster for large states. Thus the sampling rate needed may depend on the accuracy desired when controlling to zero as well as on the rough size of the initial states, and this fact is captured in the following definition. (The "s" in "s-stabilizing" is for "sampling".)*

Definition I.3: The feedback $k : \mathbb{R}^n \to \mathbb{U}$ is said to *s-stabilize* the system (1) if for each pair

$$0 \,<\, r \,<\, R$$

there exist $M = M(R) > 0$, $\delta = \delta(r, R) > 0$, and $T = T(r, R) > 0$ such that, for every partition π with $d(\pi) < \delta$

*A more cumbersome but descriptive notation would be "clss-stabilizing", for stabilization under "closed-loop system sampling" and for any initial state x_0 such that $|x_0| \leq R$, the π-trajectory $x(\cdot)$ of (2) starting from x_0 is well-defined and it holds that:

1. (**uniform attractiveness**) $|x(t)| \leq r \quad \forall\, t \geq T$;
2. (**overshoot boundedness**) $|x(t)| \leq M(R) \quad \forall\, t \geq 0$;
3. (**Lyapunov stability**) $\lim_{R\downarrow 0} M(R) = 0$.

Remark I.4: As mentioned in Section VI, if a continuous feedback k stabilizes the system (1) in the usual sense (namely, it makes the origin of (2) globally asymptotically stable), then it also s-stabilizes. In this sense, the present notion generalizes the classical notion of stabilization. □

We next recall the definition of (global, null-) asymptotic controllability. By a *control* we mean a measurable function $u : [0, +\infty) \to \mathbb{U}$ which is locally essentially bounded (meaning that, for each $T > 0$ there is some compact subset $\mathbb{U}^T \subseteq \mathbb{U}$ so that $u(t) \in \mathbb{U}^T$ for a.a. $t \in [0, T]$). In general, we use the notation $x(t; x_0, u)$ to denote the solution of (1) at time $t \geq 0$, with initial condition x_0 and control u. The expression $x(t; x_0, u)$ is defined on some maximal interval $[0, t_{\max}(x_0, u))$.

Definition I.5: The system (1) is *asymptotically controllable* if:

1. (**attractiveness**) for each $x_0 \in \mathbb{R}^n$ there exists some control u such that the trajectory $x(t) = x(t; x_0, u)$ is defined for all $t \geq 0$ and $x(t) \to 0$ as $t \to +\infty$;
2. (**Lyapunov stability**) for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x_0 \in \mathbb{R}^n$ with $|x_0| < \delta$ there is a control u as in 1. such that $|x(t)| < \varepsilon$ for all $t \geq 0$;
3. (**bounded controls**) there are a neighborhood $\mathbb{X}_0$ of 0 in $\mathbb{R}^n$, and a compact subset $\mathbb{U}_0$ of $\mathbb{U}$ such that, if the initial state x_0 in 2. satisfies also $x_0 \in \mathbb{X}_0$, then the control in 2. can be chosen with $u(t) \in \mathbb{U}_0$ for almost all t.

This is a natural generalization to control systems of the concept of uniform asymptotic stability of solutions of differential equations. The last property – which is not part of the standard definition of asymptotic controllability given in textbooks, e.g. [25] – is introduced here for technical reasons, and it has the effect of ruling out the case in which the only way to control to zero is by using controls that must "go to infinity" as the state approaches the origin. Our main result is as follows.

Theorem 1: The system (1) is asymptotically controllable if and only if it admits an s-stabilizing feedback.

One implication is trivial: existence of an s-stabilizing feedback is easily seen to imply asymptotic controllability. Note that the bounded overshoot property, together with the fact that k is locally bounded, insures that the control applied (namely, a piecewise constant control which switches at the "sampling times" in the partition) is bounded. The attractiveness property holds by iteratively controlling to balls of small radius and using the overshoot and stability estimate to insure convergence to the origin. Finally, the Lyapunov stability property holds by construction.

The interesting implication is the converse, namely the construction of the feedback law. The main ingredients in this construction are: (a) the notion of control-Lyapunov function (called just "Lyapunov function" for a control system in [25]; see also [17], [18]), (b) methods of nonsmooth analysis, and (c) techniques from positional differential games. We review these ingredients in the next section, and then develop further technical results; latter sections contain the proof as well as a robustness result.

II. Some Preliminaries

We start with a known characterization of asymptotic controllability in terms of control-Lyapunov functions. Given a function $V : \mathbb{R}^n \to \mathbb{R}$ and a vector $v \in \mathbb{R}^n$, the *lower directional derivative of V in the direction of v* is

$$DV(x; v) := \liminf_{\substack{t \downarrow 0 \\ v' \to v}} \frac{1}{t} \left(V(x + tv') - V(x) \right).$$

The function $v \mapsto DV(x; v)$ is lower semicontinuous. For a set $F \subseteq \mathbb{R}^n$, co$F$ denotes its convex hull.

Definition II.1: A *control-Lyapunov pair* for the system (1) consists of two continuous functions $V, W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that the following properties hold:

1. (**positive definiteness**) $V(x) > 0$ and $W(x) > 0$ for all $x \neq 0$, and $V(0) = 0$;
2. (**properness**) the set $\{x \mid V(x) \leq \beta\}$ is bounded for each β;
3. (**infinitesimal decrease**) for each bounded subset $G \subseteq \mathbb{R}^n$ there is some compact subset $\mathbb{U}_0 \subseteq \mathbb{U}$ such that
$$\min_{v \in \mathrm{co} f(x, \mathbb{U}_0)} DV(x; v) \leq -W(x) \tag{4}$$
for every $x \in G$.

If V is part of a control-Lyapunov pair (V, W), it is a *control-Lyapunov function (clf)*.

It was shown in [23] that asymptotic controllability is equivalent to the existence of a pair of functions (V, W) which satisfy the properties given above, except that property 3 is expressed in an apparently weaker fashion, namely, by means of derivatives along trajectories (corresponding to relaxed controls). In [28] it was observed that in fact one can reformulate the definition in the above terms, so we obtain as follows.

Theorem 2: The system (1) is asymptotically controllable if and only if it admits a control-Lyapunov function. ■

Observe that when the function V is smooth, condition (4) can be written in the more familiar form found in the literature, namely:

$$\min_{u \in \mathbb{U}_0} \langle \nabla V(x), f(x, u) \rangle \leq -W(x). \tag{5}$$

In contrast to the situation with stability of (non-controlled) differential equations, a system may be asymptotically controllable system and yet there may not exist any possible smooth clf V. In other words, there is no analogue of the classical theorems due to Massera and Kurzweil. This issue is intimately related to that of existence of continuous feedback, via what is known as Artstein's Theorem (cf. [2], [17], [18], [24]), which asserts that existence of a differentiable V

is equivalent, for systems affine in controls, to there being a stabilizing regular feedback. Nevertheless, it is possible to reinterpret the condition (5) in such a manner that relation (5) does hold in general, namely by using a suitable generalization of the gradient. Specifically, we will remark later that we may use the proximal subgradients of V at x instead of $\nabla V(x)$, replacing (5) by:

$$\min_{u\in\mathbb{U}_0} \langle \zeta, f(x,u)\rangle \;\le\; -W(x) \quad \text{for every } \zeta\in\partial_{\mathrm{P}}V(x)\,, \qquad (6)$$

where ζ and $\partial_{\mathrm{P}}V(x)$ are the proximal subgradients and the subdifferential, respectively, of the function V at the point x. The use of proximal subgradients as substitutes for the gradient for a nondifferentiable function plays a central role in our construction of feedback. The concept was originally developed in nonsmooth analysis for the study of optimization problems, see [4].

The relation (6) says that V is a "proximal supersolution" of the corresponding Hamilton-Jacobi equation, which via the results in [6] is known to be equivalent to the statement that locally V is a *viscosity supersolution* (cf. [12], [14]) of the same equation. On the other hand, the relation (4) says that V is an upper minimax solution of the same equation ([29]). The coincidence of these two solution concepts reflects the deep and intrinsic connection between invariance properties of function V with respect to trajectories of the control system (1) and the characterization of these properties in terms of proximal subgradients of V. For more discussion on these aspects of viscosity and other generalized solution concepts of first-order PDEs, and their connection to nonsmooth analysis, the reader is referred to [6] and the book [29].

Finally, we rely on methods developed in the theory of positional differential games in [19]. These techniques were used together with nonsmooth analysis tools in the construction of discontinuous feedback for differential games of pursuit in [7] and games of fixed duration in [8], and these results are relevant to the construction of stabilizing feedback in our main result.

This paper is organized as follows. In Section III, we provide a self-contained exposition of proximal subgradients and their basic properties. These methods are used in Section IV for construction of "semiglobal" (that is to say, on each compact set) stabilizing feedback, and the results are extended to the global case in Section V. Finally, section VI discusses robustness issues.

III. Proximal Subgradients and Inf-Convolutions

We recall the concept of proximal subgradient, one of the basic building blocks of nonsmooth analysis.

A vector $\zeta\varepsilon\mathbb{R}^n$ is a *proximal subgradient* (respectively, *supergradient*) of the function $V: R^n \to (-\infty,+\infty]$ at x if there exists some $\sigma>0$ such that, for all y in some neighborhood of x,

$$V(y) \;\ge\; V(x) + \langle\zeta, y-x\rangle - \sigma\,|y-x|^2 \qquad (7)$$

(respectively, $V(y)\le V(x)+\langle\zeta,y-x\rangle+\sigma\,|y-x|^2$). The set of proximal subgradients of V at x (which may be empty) is denoted by $\partial_{\mathrm{P}}V(x)$ and is called the *proximal subdifferential* of V at x. If the function V is differentiable at x, then we have from the definition that the only possible subgradient of V at x is $\zeta=\nabla V(x)$, but, we note that, unless further regularity is imposed (for instance, if V is of class C^2) then $\partial_{\mathrm{P}}V(x)$ may well be empty.

There is a simple relation between proximal subgradients of V at x and the lower directional derivatives $DV(x;v)$, which follows immediately from the definitions: for any $\zeta\in\partial_{\mathrm{P}}V(x)$ and any $v\in\mathbb{R}^n$,

$$\langle\zeta, v\rangle \;\le\; DV(x;v)\,. \qquad (8)$$

We now fix a parameter $\alpha\in(0,1]$ and define the following function:

$$V_\alpha(x) \;:=\; \inf_{y\in\mathbb{R}^n}\left[V(y)+\frac{1}{2\alpha^2}\,|y-x|^2\right]. \qquad (9)$$

This is called the *inf-convolution* of the function V (with a multiple of the squared-norm). The function V_α is well-known in classical convex analysis as the "Iosida-Moreau regularization" of (convex) V. In the case of a lower semicontinuous V which is bounded below (for instance, for a clf V, which is continuous and nonnegative), the function V_α is locally Lipschitz and is an approximation of V in the sense that $\lim_{\alpha\downarrow 0} V_\alpha(x) = V(x)$; cf. [5]. In this case, the set of minimizing points y in (9) is nonempty. We choose one of them and denote it by $y_\alpha(x)$. (We are *not* asserting the existence of a regular choice; any assignment will do for our purposes.)

By definition, then,

$$\begin{aligned} V_\alpha(x) \;&=\; V(y_\alpha(x)) + \frac{1}{2\alpha^2}\,|y_\alpha(x)-x|^2 \\ &\le\; V(y)+\frac{1}{2\alpha^2}\,|y-x|^2 \end{aligned} \qquad (10)$$

for all $y\in\mathbb{R}^n$. The vector

$$\zeta_\alpha(x) \;:=\; \frac{x-y_\alpha(x)}{\alpha^2} \qquad (11)$$

will be of special interest; it is simultaneously a proximal subgradient of V at $y_\alpha(x)$ and a proximal *super*gradient of V_α at x, as we discuss next.

Lemma III.1: For any $x\in\mathbb{R}^n$,

$$\zeta_\alpha(x) \;\in\; \partial_{\mathrm{P}}V(y_\alpha(x))\,. \qquad (12)$$

Proof: We rearrange terms in (10) and obtain, for any $y\in\mathbb{R}^n$,

$$V(y) \;\ge\; V(y_\alpha(x)) + \langle\zeta_\alpha(x), y-y_\alpha(x)\rangle - \frac{1}{2\alpha^2}\,|y-y_\alpha(x)|^2\,,$$

which implies (12), by definition of subgradients. ■

Lemma III.2: For any $x\in\mathbb{R}^n$, $\zeta_\alpha(x)$ is a proximal supergradient of V_α at x.

Proof: The assertion will be shown if we prove that, for every $x,y\in\mathbb{R}^n$,

$$V_\alpha(y) \;\le\; V_\alpha(x) + \langle\zeta_\alpha(x), y-x\rangle + \frac{1}{2\alpha^2}\,|y-x|^2\,. \qquad (13)$$

By definition, we have

$$\begin{aligned} V_\alpha(y) &\leq V(y_\alpha(x)) + \frac{1}{2\alpha^2}\left|y_\alpha(x)-y\right|^2 \\ V_\alpha(x) &= V(y_\alpha(x)) + \frac{1}{2\alpha^2}\left|y_\alpha(x)-x\right|^2 . \end{aligned}$$

Subtracting and rearranging quadratic terms, we obtain (13). ■

We deduce from Equation (13) that, for any $\tau \in \mathbb{R}^1$ and any $v \in \mathbb{R}^n$,

$$V_\alpha(x+\tau v) \;\leq\; V_\alpha(x) + \tau\langle\zeta_\alpha(x), v\rangle + \frac{\tau^2\left|v\right|^2}{2\alpha^2}. \tag{14}$$

This plays a role analogous to that of Taylor expansions for evaluating increments of the function V_α.

We introduce some additional notations and obtain several estimates which are used later. In general, we use B_R to denote the closed ball of radius R. We assume that the function V is a clf as in Definition II.1. For each $R>0$ we consider the numbers

$$\beta(R) := \max_{|x|\leq R} V(x) \quad \text{and} \quad \gamma(R) := \min_{|x|\geq R} V(x)$$

as well as the sets

$$G_R := \left\{ x \;\middle|\; V(x) \leq \frac{1}{2}\gamma(R) \right\},$$

$$G_R^\alpha := \left\{ x \;\middle|\; V_\alpha(x) \leq \frac{1}{2}\gamma(R) \right\},$$

and

$$\rho(R) := \max\left\{\rho \mid B_\rho \subseteq G_R\right\}.$$

It is easy to verify that the following relations hold:

$$\beta(R) \geq \gamma(R) > 0\,, \quad \rho(R) > 0 \quad \forall\, R > 0\,, \tag{15}$$

$$\lim_{R\downarrow 0}\beta(R) = \lim_{R\downarrow 0}\rho(R) = 0\,, \tag{16}$$

$$\lim_{R\to\infty}\gamma(R) = \lim_{R\to\infty}\rho(R) = \infty\,. \tag{17}$$

Finally, the function

$$\omega_R(\delta) := \max\left\{ V(x) - V(x') \;\middle|\; |x-x'| \leq \delta,\, x, x' \in B_{R+\sqrt{2\beta(R)}} \right\}$$

is the modulus of continuity of the function V on the ball $B_{R+\sqrt{2\beta(R)}}$.

The next Lemma provides an upper estimate of the distance between x and $y_\alpha(x)$.

Lemma III.3: For any $x \in B_R$, $|y_\alpha(x) - x| \leq \sqrt{2\beta(R)}\alpha$.

Proof: The conclusion follows from the obvious inequality

$$V_\alpha(x) \;\leq\; V(x) \tag{18}$$

as well as the inequality

$$\frac{1}{2\alpha^2}\left|y_\alpha(x)-x\right|^2 \;\leq\; V(x) - V(y_\alpha(x)) \;\leq\; V(x) \;\leq\; \beta(R)\,, \tag{19}$$

which follows from the definitions of V_α and $\beta(R)$. ■

Recall that $\alpha \in (0,1]$, so it follows from here that also

$$|x| \leq R \;\Rightarrow\; |y_\alpha(x)| \leq R + \sqrt{2\beta(R)}\,. \tag{20}$$

We next show that V is uniformly approximated by V_α on B_R.

Lemma III.4: For any $x \in B_R$,

$$V_\alpha(x) \;\leq\; V(x) \;\leq\; V_\alpha(x) + \omega_R\left(\sqrt{2\beta(R)}\alpha\right). \tag{21}$$

Proof: The first inequality is just (18). To prove the second one, we use the obvious relation

$$V_\alpha(x) \;\geq\; V(y_\alpha(x))\,.$$

By the estimate in Lemma III.3,

$$V(y_\alpha(x)) \;\geq\; V(x) - \omega_R\left(\sqrt{2\beta(R)}\alpha\right).$$

Equation (21) follows from these two inequalities. ■

We use later the inclusions

$$B_{\rho(R)} \subseteq G_R \subseteq G_R^\alpha\,, \tag{22}$$

which are valid for all $R>0$ and $\alpha>0$ and which follow directly from the definition of ρ and (18).

Lemma III.5: For any $R>0$ and α satisfying

$$\omega_R\left(\sqrt{2\beta(R)}\alpha\right) \;<\; \frac{1}{2}\gamma(R) \tag{23}$$

we have

$$G_R^\alpha \;\subseteq\; \operatorname{int} B_R\,. \tag{24}$$

Proof: Let $x \in G_R^\alpha$. By definition of G_R^α, $V_\alpha(x) \leq \frac{1}{2}\gamma(R)$, so the fact that $V_\alpha(x)$ equals the expression in (10) implies that

$$V(y_\alpha(x)) \leq \frac{1}{2}\gamma(R) \quad \text{and} \quad \frac{1}{2\alpha^2}\left|y_\alpha(x)-x\right|^2 \leq \frac{1}{2}\gamma(R)\,.$$

It follows that $\underline{y_\alpha(x)} \in B_R$ by definition of $\gamma(R)$ and $|y_\alpha(x)-x| \leq \sqrt{2\beta(R)}\alpha$ because of (15). Then

$$V(x) \;\leq\; V(y_\alpha(x)) + \omega_R\left(\sqrt{2\beta(R)}\alpha\right) \;<\; \gamma(R)\,,$$

and this implies that $|x| < R$ by definition of $\gamma(R)$. ■

IV. Semi-Global Practical Stabilization

From now on, we assume that an asymptotically controllable system (1) has been given. Applying Theorem 2, we pick a Lyapunov pair (V, W). Choose any $0 < r < R$. We will construct a feedback that sends all states in the ball of radius R ("semiglobal" stabilization) into the ball of radius r ("practical" stability, as we do not yet ask that states converge to zero). By the definition of Lyapunov pair, there is some compact subset $\mathbb{U}_0 \subseteq \mathbb{U}$ so that property (4) holds for every $x \in B_{R+\sqrt{2\beta(R)}}$. Because of the relation (8) between proximal subgradients and lower directional derivatives, we have that also condition (6) holds for the Lyapunov pair, for all $x \in B_{R+\sqrt{2\beta(R)}}$.

Pick any $\alpha \in (0,1]$. In terms of the vectors $\zeta_\alpha(x)$ introduced in Equation (11), we define a function $k_\nu : B_R \to \mathbb{U}_0$ by letting $k_\nu(x)$ be a pointwise minimizer of $\langle \zeta_\alpha(x), f(x,u)\rangle$:

$$\langle \zeta_\alpha(x), f(x, k_\nu(x))\rangle \;=\; \min_{u\in\mathbb{U}_0} \langle \zeta_\alpha(x), f(x,u)\rangle\,. \tag{25}$$

The choice $x \mapsto k_\nu(x)$ is not required to have any particular regularity properties. We use the subscript $\nu = (\alpha, r, R)$ to emphasize the dependence of the function k on the particular parameters (which represent respectively the "degree of smoothing" of V that is used in its construction, the radius of the ball to which we are controlling, and the radius of the ball on which the feedback will be effective). The next theorem says that for any fixed r, R, we can choose arbitrarily small $\alpha > 0$ and then $\delta > 0$ such that the set G_R^α is invariant with respect to any π-trajectory of

$$\dot{x} \;=\; f(x, k_\nu(x))\,, \tag{26}$$

and that $x(t)$ enters and stays in B_r for all large t, provided that the diameter of the partition π satisfies $d(\pi) \le \delta$.

Theorem 3: Let V be a clf. Then, for any $0 < r < R$ there are $\alpha_0 = \alpha_0(r,R)$ and $T = T(r,R)$ such that, for any $\alpha \in (0, \alpha_0)$ there exists $\delta > 0$ such that for any $x_0 \in G_R^\alpha$ and any partition π with $d(\pi) \le \delta$, the π-trajectory $x(\cdot)$ of (26) starting at x_0 must satisfy:

$$x(t) \in G_R^\alpha\,, \quad \forall t \ge 0 \tag{27}$$

and

$$x(t) \in B_r\,, \quad \forall t \ge T\,. \tag{28}$$

Observe that, because of (17) and (22), for every $R' > 0$ there is some $R > 0$ such that $B_{R'} \subseteq G_R^\alpha$. Thus the Theorem says that there is a feedback that steers every state of $B_{R'}$ into the neighborhood B_r; in this sense the result is semiglobal. The proof of the Theorem will take the rest of this section and will be based on a sequence of lemmas. The main idea behind the proof is to use the new function V_α (for sufficiently small α) as a Lyapunov function and to use the "Taylor expansion" formula (14) to estimate the variations of V_α along π-trajectories.

By continuity of $f(x,u)$ and the local Lipschitz property assumed, we know that there are some constants ℓ, m such that

$$|f(x,u) - f(x',u)| \le \ell\,|x - x'|\,, \quad |f(x,u)| \le m \tag{29}$$

for all $x, x' \in B_R$ and all $u \in \mathbb{U}_0$. Let

$$\Delta \;:=\; \frac{1}{3}\,\min\left\{ W(y) \;\middle|\; \frac{1}{2}\rho(r) \le |y| \le R + \sqrt{2\beta(R)} \right\}.$$

Note that $\Delta > 0$, due to the positivity of W for $x \ne 0$.

Lemma IV.1: Let $\alpha \in (0,1]$ satisfy

$$\sqrt{2\beta(R)}\alpha < \frac{1}{2}\rho(r)\,, \quad 2\ell\,\omega_R\left(\sqrt{2\beta(R)}\alpha\right) < \Delta\,. \tag{30}$$

Then, for any $x \in B_R \setminus B_{\rho(r)}$ it holds that

$$\langle \zeta_\alpha(x), f(x, k_\nu(x))\rangle \;\le\; -2\Delta\,. \tag{31}$$

Proof: It follows from (25) and (29) that

$$\begin{aligned}&\langle \zeta_\alpha(x), f(x, k_\nu(x))\rangle \\ &\quad\le\; \min_{u\in\mathbb{U}_0} \langle \zeta_\alpha(x), f(y_\alpha(x), u)\rangle \;+\; \ell\,|\zeta_\alpha(x)|\,|y_\alpha(x) - x|\,.\end{aligned}$$

But in view of Equation (12), and condition (6), applied at the point $y_\alpha(x) \in B_{R+\sqrt{2\beta(R)}}$ (recall (20)), we have that the first term in the right-hand side is upper bounded by $-W(y_\alpha(x))$. Regarding the second term, note that the estimate

$$\begin{aligned}|\zeta_\alpha(x)|\,|y_\alpha(x) - x| \;&\le\; 2(V(x) - V(y_\alpha(x))) \\ &\le\; 2\,\omega_R\left(\sqrt{2\beta(R)}\alpha\right)\end{aligned}$$

holds, by the definition of $\zeta_\alpha(x)$, the first inequality in Equation (19), the conclusion of Lemma III.3, (20), and the definition of ω_R. So we obtain

$$\langle \zeta_\alpha(x), f(x, k_\nu(x))\rangle \;\le\; -W(y_\alpha(x)) + 2\ell\,\omega_R\left(\sqrt{2\beta(R)}\alpha\right). \tag{32}$$

Since $\rho(r) \le |x| \le R$, we obtain from Lemma III.3, (20), and (30) that

$$\frac{1}{2}\rho(r) \le |y_\alpha(x)| \le R + \sqrt{2\beta(R)}\,.$$

This implies that $W(y_\alpha(x)) \ge 3\Delta$ and from (32) and (30) we obtain (31), as claimed. ∎

Now we consider any π-trajectory of (26) corresponding to a partition $\pi = \{t_i\}_{i\ge 0}$ with $d(\pi) \le \delta$, where δ satisfies the inequality

$$\left(\frac{\ell m\sqrt{2\beta(R)}}{\alpha} + \frac{m^2}{2\alpha^2}\right)\delta \;\le\; \Delta\,. \tag{33}$$

Lemma IV.2: Let α, δ satisfy (23), (30), and (33), and assume that for some index i it is the case that $x(t_i) \in G_R^\alpha \setminus B_{\rho(r)}$. Then

$$V_\alpha(x(t)) - V_\alpha(x(t_i)) \;\le\; -\Delta(t - t_i) \tag{34}$$

for all $t \in [t_i, t_{i+1}]$. In particular, $x(t) \in G_R^\alpha$ for all such t.

Proof: Denote $x_i := x(t_i)$ and consider the largest $\bar{t} \in (t_i, t_{i+1}]$ such that $x(\tau) \in B_R$ for all $\tau \in [t_i, \bar{t}]$. Note that such a $\bar{t}$ exists, since $x_i \in \operatorname{int} B_R$ due to (24). Pick any $t \in [t_i, \bar{t}]$. We have from (29) that

$$|x(\tau) - x_i| \;\le\; m(\tau - t_i)\,. \tag{35}$$

In general,

$$x(t) \;=\; x_i + (t - t_i) f_i\,, \tag{36}$$

where

$$f_i \;=\; \frac{1}{t - t_i}\int_{t_i}^{t} f(x(\tau), k_\nu(x_i))\,d\tau \;=\; f(x_i, k_\nu(x_i)) + \eta_i\,.$$

From (29) and (35) we obtain estimates

$$|f_i| \le m\,, \quad |\eta_i| \le \ell m(t - t_i)\,. \tag{37}$$

Now, using the "Taylor expansion" formula (14) and (36), we conclude that

$$\begin{aligned} V_\alpha(x(t)) - V_\alpha(x(t_i)) &= V_\alpha(x_i + (t-t_i)f_i) - V_\alpha(x_i) \\ &\leq (t-t_i)\langle \zeta_\alpha(x_i), f_i\rangle + \frac{1}{2\alpha^2}(t-t_i)^2\,|f_i|^2 \,. \end{aligned} \tag{38}$$

On the other hand, using (36) and (37) as well as Lemma IV.1, we have:

$$\begin{aligned} \langle \zeta_\alpha(x_i), f_i\rangle &\leq \langle \zeta_\alpha(x_i), f(x_i, k_\nu(x_i))\rangle + |\zeta_\alpha(x_i)|\,|\eta_i| \\ &\leq -2\Delta + \frac{\sqrt{2\beta(R)}}{\alpha}\ell m\delta \end{aligned}$$

which implies that

$$\begin{aligned} &V_\alpha(x(t)) - V_\alpha(x(t_i)) \\ &\quad\leq \left(-2\Delta + \frac{\sqrt{2\beta(R)}}{\alpha}\ell m\delta + \frac{1}{2\alpha^2}m^2\delta\right)(t-t_i)\,. \end{aligned}$$

This implies (34) for all $t \in (t_i, \bar{t}]$. In particular, (34) implies that $x(\bar{t}) \in \operatorname{int} B_R$, which contradicts the maximality of $\bar{t}$ unless $\bar{t} = t_{i+1}$. Therefore (34) holds for all $t \in [t_i, t_{i+1}]$. ∎

Next we establish that every π trajectory enters the ball B_r at time t_N, where N is the least integer such that $x(t_N) \in G_r$, and stays inside thereafter. To do this, we first show that there is a uniform upper bound on such times t_N. Figure 1 may help in understanding the constructions.

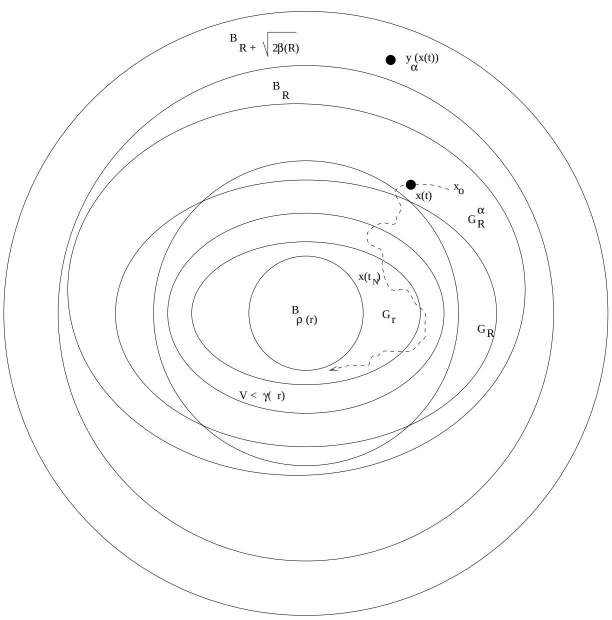

Fig. 1. *The various sets appearing in the constructions*

Lemma IV.3: Let α satisfy (23) and (30), and pick any δ so that (33) is valid. Then, for any π-trajectory $x(\cdot)$ with $d(\pi) \leq \delta$ and every $x(0) \in G_R^\alpha$, it holds that

$$t_N \;\leq\; T \;=\; \frac{\gamma(R)}{2\Delta}\,. \tag{39}$$

Proof: It follows from Lemma IV.2 and minimality of N that

$$x(t_i) \in G_R^\alpha \setminus G_r\,, \quad i = 0, 1, \ldots, N-1\,,$$

(applying Lemma IV.2 recursively and using that $B_{\rho(r)} \subseteq G_r$) and

$$0 \;\leq\; V_\alpha(x(t_N)) \;\leq\; V_\alpha(x(0)) - \Delta t_N \;\leq\; \frac{1}{2}\gamma(R) - \Delta t_N\,,$$

so (39) is valid. ∎

Lemma IV.4: Assume that, in addition to the previously imposed constraints, α and δ also satisfy the following two conditions:

$$\omega_R\left(\sqrt{2\beta(R)}\alpha\right) < \frac{1}{4}\gamma(r)\,, \tag{40}$$

(which actually implies (23)) and

$$\omega_R\,(m\delta) \;<\; \frac{1}{4}\gamma(r)\,. \tag{41}$$

Then, $x(t) \in B_r$ for all $t \geq t_N$.

Proof: If $x(t_i) \in G_r$, then

$$V_\alpha(x(t)) \leq V(x(t)) \leq V(x(t_i)) + \omega_R\,(m\delta) < \frac{3}{4}\gamma(r) \tag{42}$$

for all $t \in [t_i, t_{i+1}]$ (using (29) and (41)). Then,

$$V_\alpha(x(t_{i+1})) < \frac{3}{4}\gamma(r)\,,$$

and if $x(t_{i+1}) \notin G_r$ we can apply Lemma IV.2 to conclude that

$$V_\alpha(x(t)) < \frac{3}{4}\gamma(r) \tag{43}$$

for all $t \in [t_i, t_j]$, where j is the least integer such that $t_j > t_i$ and $x(t_j) \in G_r$ (if there is any). Starting from t_j, we may repeat the argument. We conclude that (43) holds for all $t \geq 0$. Due to Lemma III.4 and (40),

$$V(x(t)) \leq V_\alpha(x(t)) + \omega_R\left(\sqrt{2\beta(R)}\alpha\right) < \gamma(r)\,,$$

which means that $x(t) \in B_r$ for all $t \geq t_N$, as claimed. ∎

To conclude the proof of Theorem 3, let α_0 be the supremum of the set of all $\alpha > 0$ which satisfy conditions (30) and (40). Then, for any $\alpha \in (0, \alpha_0)$ we can choose δ satisfying (33) and (41), so that, for each partition with $d(\pi) \leq \delta$ and every π-trajectory starting from a state in G_R^α, the inclusions (27) and (28) hold. ∎

V. Proof of Global Result

We now prove Theorem 1. The idea is to partition the state space $\mathbb{R}^n$ into a number of "spherical shells" (more precisely, sets built out of sublevel sets of the functions V_α obtained when smoothing-out the original control-Lyapunov function) and to use a suitable feedback, constructed as in the semiglobal Theorem 3, in each shell.

We assume given an asymptotically controllable system, and a clf V for it, and we pick an arbitrary $R_0 > 0$. There is then a sequence $\{R_j\}_{j=-\infty}^{+\infty}$ satisfying

$$2R_j \;\leq\; \rho(R_{j+1})\,, \quad j = 0, \pm1, \pm2, \ldots$$

(just define the R_j's inductively for $j=1,2,\ldots$ and for $j=-1,-2,\ldots$; this is possible because of (17)). We also denote $r_j := \frac{1}{2}\rho(R_{j-1})$ for all j. We have that, for each integer j,

$$\rho(R_j) < R_j < 2R_j < \rho(R_{j+1}) \tag{44}$$

and

$$\lim_{j\to-\infty} R_j = 0\,,\quad \lim_{j\to+\infty} R_j = \infty\,. \tag{45}$$

Consider any integer j. For the pair (r_j, R_j), Theorem 3 provides the existence of numbers $\alpha_j > 0$, $\delta_j > 0$, and $T_j > 0$, and a map $k_j : B_{R_j} \to \mathbb{U}_j$, $k_j := k_{(\alpha_j, r_j, R_j)}$, such that $G_{R_j}^{\alpha_j}$ is invariant with respect to all π-trajectories of (26) when $k_\nu = k_j$ and $d(\pi) \le \delta_j$, and for each such trajectory it holds that

$$|x(t)| \,\le\, r_j\,,\quad \forall t \ge T_j\,. \tag{46}$$

Recall that in the construction of k_j, we used the fact that there is some compact subset $\mathbb{U}_0 \subseteq \mathbb{U}$, to be called here $\mathbb{U}_j$ to distinguish the sets used for the different indices j, so that condition (6) holds for the Lyapunov pair, for all $x \in B_{R_j+\sqrt{2\beta(R_j)}}$. Since the R_j form an increasing sequence, and since if the min in condition (6) also holds if we enlarge $\mathbb{U}_0$, we may, and will, assume that the $\mathbb{U}_j = \mathbb{U}_0$ for all $j<0$ and that $\mathbb{U}_j \subseteq \mathbb{U}_{j+1}$ for all $j \ge 0$. In Equation (29) we picked a bound m on the values of $|f(x,u)|$ for $x \in B_{R_j}$ and all $u \in \mathbb{U}_j$; we call this bound m_j to emphasize the dependence on j, and observe that $m_j \le m_{j+1}$ for all j, because of the monotonicity of the sets $\mathbb{U}_j$ and B_{R_j}.

Finally, we introduce the sets on which we will use the different feedbacks k_j:

$$H_j \,:=\, G_{R_{j+1}}^{\alpha_{j+1}} \setminus G_{R_j}^{\alpha_j}\,.$$

Lemma V.1: For each $i \ne j$, $H_i \bigcap H_j = \emptyset$, and

$$\mathbb{R}^n \setminus \{0\} \,=\, \bigcup_{j=-\infty}^{+\infty} H_j\,. \tag{47}$$

Proof: We know from (44), (22), and (24) that

$$G_{R_j}^{\alpha_j} \subseteq B_{R_j} \subseteq B_{2R_j} \subseteq G_{R_{j+1}}^{\alpha_{j+1}} \tag{48}$$

for all j. This implies that $H_i \subseteq G_{R_j}^{\alpha_j}$ and $H_i \bigcap H_j = \emptyset$ whenever $i<j$. For any two integers $M<N$, we obtain from (48) that

$$\{x \mid R_M \le |x| \le R_N\} \,\subseteq\, \bigcup_{j=M}^{N} H_j \,=\, G_{R_{N+1}}^{\alpha_{N+1}} \setminus G_{R_M}^{\alpha_M}\,.$$

The first inclusion together with (45) implies (47) when taking $M \to -\infty$ and $N \to +\infty$. ■

We use later the inclusion

$$B_{r_j} \,\subseteq\, \operatorname{int} G_{R_{j-1}}^{\alpha_{j-1}} \tag{49}$$

which follows from (22) and the definition of r_j.

Since the sets H_j plus the origin constitute a partition of the state space, we may define a map $k: \mathbb{R} \to \mathbb{U}$ by means of the rule

$$k(x) \,:=\, k_j(x)\,,\quad \forall\, x \in H_{j-1} \tag{50}$$

for each integer j, and $k(0) = u_0$, where u_0 is any fixed element of $\mathbb{U}_0$. To complete the proof of the Theorem, we need to show that this feedback is s-stabilizing.

Lemma V.2: The set $G_{R_j}^{\alpha_j}$ is invariant with respect to π-trajectories of (2), when (50) is used and

$$d(\pi) \,<\, \min\left\{\delta_j, \frac{R_{j-1}}{m_j}\right\}\,. \tag{51}$$

Proof: Consider any π-trajectory starting at a state in $G_{R_j}^{\alpha_j}$, where π satisfies (51). It is enough to show that $x(t) \in G_{R_j}^{\alpha_j}$ for all $t \in [t_i, t_{i+1}]$ if $x(t_i) \in G_{R_j}^{\alpha_j}$. Pick any such i. There are three possibilities which must be treated separately: (a) $x(t_i) \in H_{j-1}$, (b) $x(t_i) \in H_{\ell-1}$ for some $\ell \le j-1$, and (c) $x(t_i) = 0$. In the first case, $k(x) = k_j(x)$ is known to leave $G_{R_j}^{\alpha_j}$ invariant, so there is nothing to prove.

Assume now that we are in case (b). Observe that the partition π may not be fine enough to guarantee that $G_{R_\ell}^{\alpha_\ell}$ is invariant under the feedback k_ℓ, so we need to argue in a different way, namely that the trajectory cannot go too far because the sampling time is small. Since $x(t_i) \in H_{\ell-1} \subset G_{R_\ell}^{\alpha_\ell} \subset B_{R_\ell}$, $|x(t_i)| < R_\ell$. Pick any $\bar{t} \in [t_i, t_{i+1}]$ so that $|x(t)| \le R_j$ for all $t \in [t_i, \bar{t}]$. It follows from the choice of m_j and the fact that $k(x(t_i)) \in \mathbb{U}_\ell \subseteq \mathbb{U}_j$ that $|x(t) - x(t_i)| \le m_j(t - t_i)$ for all $t \in [t_i, \bar{t}]$ and from here that

$$|x(t)| \,\le\, m_j(t-t_i) + R_\ell \,\le\, R_{j-1} + R_\ell \,\le\, 2R_{j-1}\,, \tag{52}$$

for all $t \in [t_i, \bar{t}]$. We claim that

$$|x(t)| < R_j \tag{53}$$

for all $t \in [t_i, t_{i+1}]$. Indeed, if this were not the case, then there is some $\bar{t} \in [t_i, t_{i+1}]$ so that $|x(t)| \le R_j$ for all $t \in [t_i, \bar{t}]$ and $|x(\bar{t})| = R_j$. Since also $|x(t)| \le R_j$ for all $t \in [t_i, \bar{t}]$, Equation (52) holds, which gives $R_j \le 2R_{j-1}$, contradicting (44). We conclude that (53) holds, so the argument leading to (52) can be applied with $\bar{t} = t_{i+1}$ and we conclude that the trajectory stays in $B_{2R_{j-1}}$, which is included in $G_{R_j}^{\alpha_j}$ by (48).

There remains the special case $x(t_i) = 0$. Recall that $k(0) \in \mathbb{U}_0 \subseteq \mathbb{U}_j$. Then the argument given for case (b) can be repeated, where in (52) we have 0 instead of R_ℓ; we conclude that the trajectory stays in $B_{R_{j-1}}$, so it is in $G_{R_j}^{\alpha_j}$. ■

We now prove that the feedback k is s-stabilizing. We claim that for any $0<r<R$, there exist $\delta = \delta(r,R) > 0$, $T = T(r,R) > 0$, and $M = M(R)$, such that, when (50) is used and $d(\pi) < \delta$, every π-trajectory $x(\cdot)$ of (2) with $|x(0)| \le R$ satisfies:

$$|x(t)| \le r\,,\quad \forall t \ge T$$

and

$$|x(t)| \le M\,, \quad \forall t \ge 0\,. \tag{54}$$

We start by choosing an integer $K = K(r)$ and the least integer $N = N(R)$ such that

$$G^{\alpha_K}_{R_K} \subseteq B_r \;\text{ and }\; B_R \subseteq G^{\alpha_N}_{R_N}\,, \tag{55}$$

and define

$$\delta \;:=\; \min\left\{ \min_{K\le j\le N} \delta_j\,,\; \frac{R_{K-1}}{m_N} \right\}\,,$$

$$T \;:=\; \sum_{j=K+1}^{N} T_j\,,$$

and

$$M \;:=\; R_{N(R)}\,.$$

Consider the π-trajectory starting from some $x_0 \in B_R$. We claim that at some instant $t_i \in \pi$ such that $t_i \le T$ it holds that $x(t_i) \in G^{\alpha_K}_{R_K}$. If $x_0 \notin G^{\alpha_K}_{R_K}$, then $x_0 \in H_{N'}$ for some integer $K \le N' \le N-1$. The set $G^{\alpha_{N'}}_{R_{N'}}$ is invariant, by Lemma V.2 (notice the choice of δ, consistent with what is needed there). Therefore the trajectory remains in $H_{N'}$ until it enters $G^{\alpha_{\mu-1}}_{R_{\mu-1}}$ in some interval $(t_{i-1}, t_i]$, for some $\mu < N'$. We note that $t_i \le T_{N'}$, since due to Theorem 3 if $x(\cdot)$ remains in $H_{N'}$ until moment $T_{N'}$ then

$$x(T_{N'}) \in B_{r_{N'}}\,.$$

Because of (49), this contradicts our assumption that $x(\cdot)$ remains in $H_{N'}$. If $\mu \le K+1$, we are done. Otherwise we repeat the argument. In conclusion, $x(t_i) \in G^{\alpha_K}_{R_K}$ for some $t_i \le T$, as claimed.

But then Lemma V.2 insures that $x(\cdot)$ stays in this set for all future $t \ge t_i$, which in accordance with (55) implies (54). It follows from these considerations that $x(t)$ remains in $G^{\alpha_N}_{R_N} \subseteq B_{R_N}$ for all $t \ge 0$. Thus $|x(t)| \le M(R)$ for all t, and the claim is established. We conclude that the feedback k has the attractiveness and bounded overshoot properties required by Definition I.3. To complete the proof of the Theorem, we must still verify that k has the Lyapunov stability property postulated in that definition, namely, that the function $M(R)$ just defined satisfies also

$$\lim_{R\downarrow 0} M(R) = 0\,. \tag{56}$$

It follows from the definition of $N = N(R)$ that B_R is not contained in $G^{\alpha_{N-1}}_{R_{N-1}}$. Therefore

$$\rho(R_{N-1}) < R$$

and we get from this inequality that

$$\lim_{R\downarrow 0} N(R) \;=\; -\infty\,.$$

Then (56) follows from the fact that $\lim_{j\to-\infty} R_j = 0$. ∎

VI. Robustness Properties

One of the main justifications for the use of feedback control as opposed to open-loop control lies in the robustness properties of the former. Certainly, any feedback law automatically accounts for sudden changes in states. However, it is not necessarily the case that a given feedback law, especially a discontinuous one, will stabilize a system which is not identical to the model assumed for its design.

We show in this section that the s-stabilizing feedback control k defined in (50) is indeed insensitive to small perturbations in the system dynamics. In particular, we establish the existence of a continuous function $\chi : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $\chi(x) > 0$ for $x \ne 0$ (which depends on the system (1) being controlled), such that for any continuous function $g : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ satisfying

$$|g(x,u) - f(x,u)| \;\le\; \chi(x)\,, \quad \text{for all } \; x \in \mathbb{R}^n, u \in \mathbb{U}\,, \tag{57}$$

the same feedback k is s-stabilizing also for the control system

$$\dot{x} \;=\; g(x,u)\,. \tag{58}$$

(Note that this result also demonstrates, in view of Theorem 1, that the asymptotic controllability property is structurally stable with respect to perturbations g of the system (1) which satisfy (57).) Actually, we prove an even more general result than the s-stabilization of (58), expressed in terms of differential inclusions.

Let us consider π-trajectories of the differential inclusion

$$\dot{x} \;\in\; f(x,k(x)) + B_{\chi(x)} \tag{59}$$

where $k : \mathbb{R}^n \to \mathbb{U}$ is an arbitrary (not necessarily continuous) function and $\chi : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is continuous. For a given partition π of the interval $[0,+\infty)$, a (complete) π-trajectory $x(\cdot)$ of (59) is any absolutely continuous function defined on $[0,\infty)$ and satisfying recursively the differential inclusion

$$\dot{x}(t) \;\in\; f(x(t),k(x(t_i))) + B_{\chi(x(t))}\,, \quad \text{a.a. } t \in [t_i, t_{i+1}] \tag{60}$$

for all i.

Definition VI.1: The feedback $k : \mathbb{R}^n \to \mathbb{U}$ is said to *robustly* s-stabilize the system (1) if there exists some continuous function $\chi : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $\chi(x) > 0$ for $x \ne 0$, such that for every pair $0 < r < R$ there exist $\delta = \delta(r,R) > 0$, $T = T(r,R) > 0$, and $M = M(R)$, such that, for every partition π with $d(\pi) < \delta$, and every π-trajectory $x(\cdot)$ of (59) with $|x(0)| \le R$, all conditions 1., 2., 3. in Definition I.3 hold.

The concept of robust s-stabilizing feedback k formulated in terms of solutions of differential inclusions is closely connected with robustness properties of the feedback k in the sense of stabilization of the perturbed system

$$\dot{x} \;=\; f(x,k(x)) + w(t) \tag{61}$$

where $w : [0,+\infty) \to \mathbb{R}^n$ is an integrable function (which we call a "disturbance"). For the partition π of $[0,+\infty)$ the

π-trajectory of (61) starting from x_0 is the absolutely continuous function recursively defined on intervals $[t_i, t_{i+1}]$ as solutions of the following differential equation:

$$\dot{x}(t) \;=\; f(x(t), k(x(t_i))) + w(t)\,, \quad t \in [t_i, t_{i+1}]\,. \qquad (62)$$

Remark VI.2: There is an obvious relation between π-trajectories of the differential inclusion (59) and π-trajectories of the perturbed system (61): $x(\cdot)$ is a π-trajectory of (59) if and only if there is a measurable disturbance $w(\cdot)$ such that

$$|w(t)| \;\le\; \chi(x(t))\,, \quad \text{a.a. } t \ge 0 \qquad (63)$$

and $x(\cdot)$ is a π-trajectories of (61). □

We show that a robustly stabilizing feedback provides stabilization of the perturbed system (62) in the following sense.

Definition VI.3: The feedback $k : \mathbb{R}^n \to \mathbb{U}$ is said to be *robustly practically stabilizing* for (61) if for every pair $0 < r < R$ there exist $\delta > 0$, $T > 0$, and $\chi_0 > 0$, such that, for any disturbance $w(\cdot)$ satisfying

$$|w(t)| \;\le\; \chi_0\,, \quad \text{a.a. } t \ge 0\,, \qquad (64)$$

and any partition π such that $d(\pi) < \delta$, every π-trajectory of (62) with $|x(0)| \le R$ is well-defined on $[0, \infty)$ and conditions 1., 2., 3. in Definition I.3 hold. □

We have the following relation between the two concepts of robustness.

Proposition VI.4: If the feedback k is robustly s-stabilizing, then it is robustly practically stabilizing for (61).

Proof: Let us fix any pair $0 < r < R$ and choose r' such that

$$2r' < r,\; M(2r') < r\,.$$

We define constants

$$\chi_0 := \min\left\{\frac{1}{2}\chi(x) \mid r' \le |x| \le M(R)\right\}$$

and

$$\delta_0 := \min\left\{\delta(r', R), \frac{r'}{2m}\right\}\,,$$

where $m := \max\{|f(x,u)| + \chi(x) \mid \; |x| \le r\}$, and where the functions χ and δ are as in Definition VI.1. Then for every partition π with $d(\pi) < \delta$, any π-trajectory $x(\cdot)$ of the perturbed system(65) with the disturbance $w(\cdot)$ satisfying (64) and $|x(0)| \le R$ is a π-trajectory of the differential inclusion (59) as long as $r' \le |x(t)| \le M(R)$. Since k is robustly s-stabilizing, $|x(t)|$ stays bounded by $M(R)$ until the first time $t_i \le T(r', R)$, $t_i \in \pi$, such that $|x(t_i)| \le r'$. Assume that after this moment, $x(t_l)$, $l = i+1, \ldots, j-1$ belongs to the ball $B_{r'}$, but $x(t_j) \notin B_{r'}$. Due to the choice of δ_0 we have that $x(t) \in B_{2r'}$ for $t \in [t_i, t_j]$. Then $r' \le |x(t_j)| \le 2r'$ and $x(\cdot)$ again is a π-trajectory of the differential inclusion (59) driven by the feedback k until the next moment $t_k > t_j$ such that $|x(t_k)| < r'$. It follows that

$$|x(t)| \le M(2r') < r \;\text{ for } t \in [t_j, t_k]\,.$$

Finally, we can repeat this argument to obtain that $|x(t)| \le r$ for all $t > T(r', R)$, which proves the Proposition. (Observe that since $x(\cdot)$ is a solution of an ordinary differential equation, boundedness implies global existence.) ■

The main result of this Section is as follows:

Theorem 4: Let the system (1) be asymptotically controllable. Then the feedback k defined in (50) is robustly s-stabilizing.

We base the proof of this Theorem on the proof of the following semiglobal practical robust stabilization result, in the same manner that the proof of Theorem 1 is based on Theorem 3.

Theorem 5: Let V be a clf. Then, for every $0 < r < R$ there exist $\alpha_0 = \alpha_0(r, R)$ and $T = T(r, R)$ such that for any $\alpha \in (0, \alpha_0)$ there exists $\chi_0 > 0$, $\delta > 0$ such that for any $x_0 \in G_R^\alpha$, any feedback k_ν defined as in (25), and any disturbance $w(\cdot)$ satisfying (64), and any partition π with $d(\pi) < \delta$, the π-trajectory $x(\cdot)$ of the perturbed system

$$\dot{x} \;=\; f(x, k_\nu(x)) + w(t) \qquad (65)$$

starting from x_0 must satisfy (27) and (28).

The proof of this Theorem follows from the proof of Theorem 3 with some minor changes. We explain just the needed changes, not repeating all details, and using the same notations as in Section IV.

We start by defining

$$m' \;:=\; \max\left\{|f(x,u)| \mid x \in B_R, u \in \mathbb{U}_0\right\}$$

and redefine the constant m now as

$$m := 2m'\,.$$

Then the upper bound χ_0 in (64) on the magnitude of the admissible disturbance is given as

$$\chi_0 \;=\; \min\left\{m', \frac{\Delta\alpha}{2\sqrt{2\beta(R)}}\right\}\,. \qquad (66)$$

For this choice of χ_0 we have that the inequalities in (29) can be replaced by

$$|f(x,u) - f(x',u)| \le \ell\,|x - x'|\,, \quad |f(x,u) + w| \le m\,, \qquad (67)$$

which hold for all $x, x' \in B_R$, all $u \in \mathbb{U}_0$, and all $w \in B_{\chi_0}$. Because of (66) we have from Lemma III.3 that for all $x \in B_R$ and $w \in B_{\chi_0}$:

$$\langle \zeta_\alpha(x), w\rangle \;\le\; |\zeta_\alpha(x)|\,\chi_0 \;\le\; \frac{1}{2}\Delta\,. \qquad (68)$$

Now we analyze the behavior of any π-trajectory of the perturbed system (65) assuming that the disturbance satisfies (64). In place of Lemma IV.2, we substitute the following:

Lemma VI.5: Let α, δ satisfy (23), (30), and (33),, and assume that for some index i it is the case that $x(t_i) \in G_R^\alpha \setminus B_{\rho(r)}$. Then

$$V_\alpha(x(t)) - V_\alpha(x(t_i)) \;\le\; -\frac{1}{2}\Delta(t - t_i) \qquad (69)$$

for all $t \in [t_i, t_{i+1}]$. In particular, $x(t) \in G_R^\alpha$ for all such t.

The proof of this Lemma follows along the lines of the proof of Lemma IV.2, but with one important modification due to the definition (62) of $x(\cdot)$ as a π-trajectory of (65). Namely, f_i in (36) is defined now instead as follows:

$$\begin{aligned} f_i &= \frac{1}{t-t_i}\int_{t_i}^{t} [f(x(\tau), k_\nu(x_i)) + w(\tau)]\, d\tau \\ &= f(x_i, k_\nu(x_i)) + \frac{1}{t-t_i}\int_{t_i}^{t} w(\tau)\, d\tau + \eta_i\,. \end{aligned}$$

Note that due to (67) the estimate (37) holds again in this case. This implies, because of (68):

$$\langle \zeta_\alpha(x_i), f_i\rangle \le \langle \zeta_\alpha(x_i), f(x_i, k_\nu(x_i))\rangle + |\zeta_\alpha(x_i)|\,|\chi_0| + |\zeta_\alpha(x_i)|\,|\eta_i|$$

and therefore, due to (31), (37), and (68):

$$\langle \zeta_\alpha(x_i), f_i\rangle \le -\frac{3}{2}\Delta + \frac{\sqrt{2\beta(R)}}{\alpha}\ell m\delta\,.$$

Next, applying the "Taylor expansion" formula (38), we obtain:

$$V_\alpha(x(t)) - V_\alpha(x(t_i)) \le \left(-\frac{3}{2}\Delta + \frac{\sqrt{2\beta(R)}}{\alpha}\ell m\delta + \frac{1}{2\alpha^2}m^2\delta\right)(t-t_i)\,.$$

In view of the choice of α and δ, we obtain that (69) holds for all $t \in (t_i, \bar t]$. In particular, (69) implies that $x(\bar t) \in \operatorname{int} B_R$, which contradicts the maximality of $\bar t$ unless $\bar t = t_{i+1}$. Therefore (69) holds for all $t \in [t_i, t_{i+1}]$.

The decreasing property for V_α along every π-trajectory of the perturbed system (65) lets us establish that every π-trajectory enters the ball B_r at time t_N, where N is the least integer such that $x(t_N) \in G_r$.

Lemma VI.6: Let α satisfy (23) and (30), and χ_0 satisfy (66). Pick $\delta > 0$ so that (33) holds. Then for every disturbance $w(\cdot)$ satisfying (64), and any π-trajectory of (65) with $d(\pi) \le \delta$ and every $x(0) \in G_R^\alpha$, it holds that

$$t_N \le T = \frac{\gamma(R)}{2\Delta}\,. \tag{70}$$

Proof: It follows from Lemma VI.5 and minimality of N that

$$x(t_i) \in G_R^\alpha \setminus G_r\,, \quad i = 0, 1, \ldots, N-1\,,$$

applying Lemma IV.2 recursively and using that $B_{\rho(r)} \subseteq G_r$ we obtain

$$0 \le V_\alpha(x(t_N)) \le V_\alpha(x(0)) - \frac{1}{2}\Delta t_N \le \frac{1}{2}((\gamma(R) - \Delta t_N)\,.$$

This shows (70). ■

Finally, note that Lemma IV.4 is valid for any π-trajectory $x(\cdot)$ of the perturbed system (65) and any disturbance $w(\cdot)$ satisfying (64), because of the choice of χ_0 in (66) and the estimates (67).

To conclude the proof of this theorem, we let α_0 be the supremum of all the $\alpha > 0$ satisfying (23) and (30). Then, for every $\alpha \in (0, \alpha_0)$ we can choose χ_0 satisfying (66) and δ satisfying (33) and (41) so that for any disturbance $w(\cdot)$ bounded by χ_0 and any π-trajectory $x(\cdot)$ of the perturbed system (65) with $d(\pi) \le \delta$ starting from G_R^α the inclusions (27) and (28) hold. ■

Proof of Theorem 4. We use the same notations as in Section V and follow closely the scheme of the proof of Theorem 1, pointing out the necessary modifications. Let

$$m_j' := \max\left\{|f(x,u)| \mid x \in B_{R_j}, u \in \mathbb{U}_j\right\}\,,$$

$$\chi_j := \min\left\{m_j'\,,\, \frac{\Delta\alpha_j}{2\sqrt{2\beta(R_j)}}\right\}\,,$$

$$m_j = 2m_j'\,.$$

Without loss of generality, we can assume that m_j is monotone increasing. We define the function

$$\chi(x) := \inf\left\{\widetilde\chi(y) + |y - x| \mid y \in \mathbb{R}^n\right\}\,,$$

where $\widetilde\chi$ is a function defined as follows: $\widetilde\chi(0) = 0$,

$$\widetilde\chi(x) = \chi_j \;\text{ for }\; x \in H_{j-1}\,. \tag{71}$$

It is obvious that the function χ so defined is Lipschitz with constant 1 in all of $\mathbb{R}^n$ and it satisfies

$$0 < \chi(x) \le \chi_j \;\text{ for }\; x \in H_{j-1}\,. \tag{72}$$

We now consider the π-trajectory $x(\cdot)$ of the differential inclusion (59). In view of Remark VI.2, $x(\cdot)$ is also a solution of the perturbed system (61) corresponding to some disturbance $w(\cdot)$ which satisfies (63).

Due to the inequality (72), we obtain that if $x(t_i) \in H_{j-1}$ for some t_i, j, and

$$\widetilde t := \sup\left\{t \ge t_i \,|\, x(t) \in H_{j-1}\right\}\,,$$

then on $[t_i, \widetilde t]$, $x(\cdot)$ is a π-trajectory of the perturbed system (65) with $\nu = \nu_j$ and $w(\cdot)$ bounded by χ_j. Note that if $x(\cdot)$ stays in $G_{R_j}^{\alpha_j}$ then $\left|x(t) - x(\widetilde t)\right| \le m_j(t - \widetilde t)$ for any $t, \widetilde t$.

This means in accordance with Theorem 5, Lemmas VI.5 and VI.6 that we have a result analogous to Lemma V.2:

Lemma VI.7: The set $G_{R_j}^{\alpha_j}$ is invariant with respect to π-trajectories of the differential inclusion (59), when (50) is used and (51) holds. □

Moreover, due to Lemma VI.6 we have that for every j there is a T_j such that for any π-trajectory of (59) starting from H_{j-1} there is a moment $t' \le T_j$ such that $x(t') \in G_{R_{j-1}}^{\alpha_{j-1}}$ if $d(\pi) \le \delta_j$. Then, to complete the proof of the fact that k is robustly s-stabilizing we need to repeat word by word the end of the proof in Section V, starting from (54). ■

Corollary VI.8: If the system(1) is asymptotically controllable then there is a continuous function $\chi : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $\chi(x) > 0$ for $x \ne 0$ such that any continuous function

$g(x, u)$ satisfying (57), any s-stabilizing feedback (50) is s-stabilizing for the system (58). □

Indeed, it is obvious that any π-trajectory of

$$\dot{x} = g(x, k(x)) \tag{73}$$

is a π-trajectory of the differential inclusion (59); thus the Corollary follows from Theorem 4.

A. Concluding Remarks

It easily follows from Theorem 4 and Proposition VI.4 that the feedback k in (50) is robust with respect to *actuator* errors. By this we mean that, for arbitrary $0 < r < R$, for all sufficiently small perturbations $e(\cdot)$ and $w(\cdot)$, all π-trajectories of

$$\dot{x}(t) = f(x(t), k(x(t)) + e(t)) + w(t)$$

with sufficiently small sampling periods $d(\pi)$ satisfy conditions 1. and 2. in Definition I.3. We leave to reader to fill the details.

On the other hand, we have not made any statements regarding *measurement* errors. Indeed, our technique is *not* robust with respect to such errors. It is possible to produce examples of systems for which every s-stabilizing state feedback will fail to stabilize if the input $k(x(t) + e(t))$ is used instead of $k(x(t))$. This question is related to the fact that feedback robust to measurement error results essentially in solutions of certain differential inclusions (see [15]). However, it is possible to provide a robust dynamic (rather than purely state) stabilizing feedback; the paper [20] deals with that issue.

It was mentioned in Section II that a continuous feedback which stabilizes in the usual sense is in particular s-stabilizing. We now make this precise. For a continuous feedback k (not necessarily Lipschitz, so there is no uniqueness of solutions), stabilizability is taken here to mean that, for each $0 < r < R$ there exist $T = T(r, R)$ and $M = M(R)$ such that for every trajectory of the closed loop system (2) with $|x(0)| \leq R$ we have conditions 1., 2., 3. in Definition I.3.

Proposition VI.9: If k is a continuous stabilizing feedback then it is s-stabilizing.

We first establish:

Lemma VI.10: Let $T(r, R)$ and $M(R)$ be as in the definition of continuous stabilizing feedback. Then there exists $\delta = \delta(r, R)$ such that, for any partition π with $d(\pi) \leq \delta$ and any π-trajectory of (2) with $|x(0)| \leq R$,

$$|x(t)| < 2M(R) \ \text{ for all } t \in [0, T] \tag{74}$$

and

$$|x(T)| < 2r\,. \tag{75}$$

Proof: Assume on the contrary that there is a sequence $\delta_k \downarrow 0$ and partitions π_k with $d(\pi_k) \leq \delta_k$, and π_k-trajectories $x_k(\cdot)$ starting from B_R such that (74) does not hold. This means that there is some sequence $t_k \in [0, T]$ such that

$$|x_k(t)| < 2M(R_i)\,\forall\, t \in [0, t_k)\,,\ |x_k(t_k)| = 2M(R)\,. \tag{76}$$

Without loss of generality, we can assume that $t_k \to \widetilde{t} \in [0, T]$ and $x_k(\cdot) \to x(\cdot)$ uniformly on $[0, T]$ (by the Arzela-Ascoli theorem, since the functions are equibounded and equicontinuous because their derivatives are estimated by an upper bound of $f(x, k(x))$ on B_R). Because of continuity of k and the fact that $\delta_k \to 0$, we conclude that $x(\cdot)$ is a trajectory of (2) with $|x(0)| \leq R$. But by (76) we have that $\left|x(\widetilde{t})\right| = 2M(R)$, which contradicts condition 2. in Definition I.3. The proof that (75) holds is analogous. ■

We now prove Proposition VI.9. We $0 < r < R$, and choose $r' > 0$ such that

$$2M(2r') < r\,,\ 2r' < r\,,$$

and

$$\delta' \ := \ \min\left\{\delta(\frac{1}{2}r', R), \delta(\frac{1}{2}r', 2r'), \frac{r'}{2m}\right\}\,,$$

where $m = \max\{f(x, k(x)) \mid |x| \leq r\}$.

Then we obtain from Lemma VI.10 that any π-trajectory of (2) with $d(\delta) \leq \delta'$ satisfies (74) and $|x(T)| \leq \frac{1}{2}r'$ for $T = T(\frac{1}{2}r', R)$. This means that for some $t_i \in \pi$ we have $|x(t_i)| < r'$ and $x(t)$ stays in $B_{r'}$ for those moments $t_i, t_{i+1}, \ldots, t_{j-1}$ while $|x(t_j)| \geq r'$. Due to the choice of δ', we have that $|x(t_j)| \leq 2r' < r$. We again apply Lemma VI.10, now for $T' = T(\frac{1}{2}r', 2r')$ and $M' = M(2r')$, to obtain that

$$|x(t)| < 2M' < r \ \text{ for all } t \in [t_j, t_j + T']\,.$$

After that, we find the first time $t_k \in \pi$ such that $t_k > t_j$ and $|x(t_k)| < r'$, and repeat the arguments above. This implies that $x(t)$ stays in B_r for all $t \geq T$, which proves the Proposition. ■

References

[1] Aubin, J.-P., and A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*, Springer-Verlag, Berlin, 1984.

[2] Artstein, Z., "Stabilization with relaxed controls," *Nonl. Anal., TMA* **7**(1983): 1163-1173.

[3] Brockett, R.W., "Asymptotic stability and feedback stabilization," in *Differential Geometric Control theory* (R.W. Brockett, R.S. Millman, and H.J. Sussmann, eds.), Birkhauser, Boston, 1983, pp. 181-191.

[4] Clarke, F.H., *Methods of Dynamic and Nonsmooth Optimization.* Volume 57 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, S.I.A.M., Philadelphia, 1989.

[5] Clarke, F.H., Yu.S. Ledyaev, and P. Wolenski, "Proximal analysis and minimization principles," *J. Math. Anal. Appl.* **196** (1995): 722-735

[6] Clarke, F.H., Yu.S. Ledyaev, R.J. Stern, and P. Wolenski, "Qualitative properties of trajectories of control systems: A survey," *J. Dynamical and Control Systems* **1**(1995): 1-48 .

[7] Clarke, F.H., Yu.S. Ledyaev, and A.I. Subbotin, "Universal feedback strategies for differential games of pursuit," *SIAM J. Control*, 1997, to appear.

[8] Clarke, F.H., Yu.S. Ledyaev, and A.I. Subbotin, "Universal feedback via proximal aiming in problems of control and differential games," Rapport CRM-2386, Centre de Recherches Mathématiques, U. de Montréal, 1994.

[9] Coron, J-M., "Global asymptotic stabilization for controllable systems without drift," *Math of Control, Signals, and Systems* **5**(1992): 295-312.

[10] Coron, J-M., "Stabilization in finite time of locally controllable systems by means of continuous time-varying feedback laws," *SIAM J. Control and Opt.* **33**(1995): 804-833.

[11] Coron, J.-M., and L. Rosier, "A relation between continuous time-varying and discontinuous feedback stabilization," *J.Math. Systems, Estimation, and Control* **4**(1994): 67-84.

[12] Crandall, M., and P.-L. Lions (1983) "Viscosity solutions of Hamilton-Jacobi equations," *Trans. Amer. Math. Soc.* **277**(1983): 1-42.

[13] Filippov, A.F., "Differential equations with discontinuous right-hand side," *Matem. Sbornik* **5**(1960): 99-127. English trans. in *Amer. Math. Translations* **42**(1964): 199-231.

[14] Fleming, W.H., and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York, 1993.

[15] Hermes, H., "Discontinuous vector fields and feedback control," in *Differential Equations and Dynamic Systems* (J.K. Hale and J.P. La Salle, eds.), Academic Press, New York, 1967.

[16] Hermes, H., "On the synthesis of stabilizing feedback control via Lie algebraic methods," *SIAM J. Control and Opt.* **18**(1980): 352-361.

[17] Isidori, A., *Nonlinear Control Systems, Third Edition*, Springer-Verlag, London, 1989.

[18] Krstic, M., I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and adaptive control design*, John Wiley & Sons, New York, 1995.

[19] Krasovskii, N.N., and A.I. Subbotin, *Positional differential games*, Nauka, Moscow, 1974 [in Russian]. French translation *Jeux differentiels*, Editions Mir, Moscou, 1979. Revised English translation *Game-Theoretical Control Problems*, Springer-Verlag, New York, 1988.

[20] Ledyaev, Yu.S., and E.D. Sontag, "A remark on robust stabilization of general asymptotically controllable systems," submitted.

[21] Ryan, E.P., "On Brockett's condition for smooth stabilizability and its necessity in a context of nonsmooth feedback," *SIAM J. Control Optim.* **32**(1994): 1597-1604.

[22] Sontag, E.D., "Nonlinear regulation: The piecewise linear approach," *IEEE Trans. Autom. Control* **AC-26**(1981): 346-358.

[23] Sontag E.D., "A Lyapunov-like characterization of asymptotic controllability," *SIAM J. Control and Opt.* **21**(1983): 462-471.

[24] Sontag, E.D., "A 'universal' construction of Artstein's theorem on nonlinear stabilization," *Systems and Control Letters*, **13**(1989): 117-123.

[25] Sontag E.D., *Mathematical Control Theory, Deterministic Finite Dimensional Systems*, Springer-Verlag, New York, 1990.

[26] Sontag E.D., "Control of systems without drift via generic loops," *IEEE Trans. Autom. Control* **40**(1995): 1210-1219.

[27] Sontag, E.D., and H.J. Sussmann, "Remarks on continuous feedback," in *Proc. IEEE Conf. Decision and Control, Albuquerque, Dec. 1980*, IEEE Publications, Piscataway, pp. 916-921.

[28] Sontag, E.D., and H.J. Sussmann, "Nonsmooth control-Lyapunov functions," *Proc. IEEE Conf. Decision and Control, New Orleans, Dec. 1995*, IEEE Publications, 1995, pp. 2799-2805.

[29] Subbotin, A.I., *Generalized Solutions of First-Order PDEs: The Dynamical Optimization Perspective*, Birkhaüser, Boston, 1995.

[30] Sussmann, H.J., "Subanalytic sets and feedback control," *J. Differential Equations* **31**(1979: 31-52.