DETECTABILITY OF NONLINEAR SYSTEMS

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Abstract

We propose a definition of detectability for nonlinear systems. This definition generalizes naturally the standard property in the linear case, and is consistent with the "input to state stability" approach to controlled stability. We provide a characterization in terms of a dissipation inequality involving storage (Lyapunov) functions.

1. Introduction

This paper proposes two notions of *detectability* for nonlinear systems and shows their equivalence. Recall (see e.g. [5], Chapter 6) that detectability (or "asymptotic observability") for linear systems with no controls $\dot{x} = Ax$, $y = Cx$ is the property that $x(t) \to 0$ for every trajectory for which the output is identically zero: $Cx(t) \equiv 0$. This property is one of the most important in systems analysis, because it holds that every state of a linear system can be driven asymptotically to zero using only output (not state) measurements (we say that in that case the system is "i/o stabilizable") if and only if the system is both detectable and state stabilizable (meaning that for each state there is a control that sends that state asymptotically to zero).

There have been many attempts to generalize this concept to not necessarily linear systems of the type $\dot{x} = f(x,u)$, $y = h(x)$. A state-space definition, which allows showing a version of "i/o stabilizable = detectable plus state stabilizable" was proposed in the paper [4]. The definition was that under zero inputs, states whose outputs are identically zero should form an asymptotically stable subsystem. Since input perturbations are not taken into account in the definition, however, this notion is too weak; it has the disadvantage of not being "robust" enough to input or measurement disturbances. On the other hand, the paper [6] gave a definition of "observability" for nonlinear systems which does naturally incorporate such robustness, since it is expressed in the language of "input to state stability" (cf. [7]). Observability was defined there, essentially, as the property that "small inputs and small outputs should imply small state trajectories" and "bounded inputs and outputs imply bounded trajectories". This generalizes the linear concept of observability in a natural manner. The usual way of understanding detectability is as the relaxation of observability that results when asking for convergence instead of precise state values. That is, estimates on trajectories are only required to be asymptotic rather than uniform in time. Thus, when combined with the definition in [4], it makes sense to define detectability as the property that states should converge to zero when inputs and outputs are zero, and in general be ultimately bounded by a bound that depends only on the magnitude of inputs and outputs. Thus *the* natural definition is to require that an estimate of the following type holds:

$$|x(t,\ \xi)| \ \le\ \max\left\{\beta(|\xi|\,,\,t),\ \gamma_1\left(\left\|u|_{[0,t]}\right\|\right),\ \gamma_2\left(\left\|y|_{[0,t]}\right\|\right)\right\}$$

for every initial state ξ.* This is entirely analogous to the definition of ISS (input to state stability), which results if we omit y from the estimate. Moreover, it is almost the same as (actually, a particular case of) the notion called "strong unboundedness observability" in [1]; in that paper, the authors were also motivated by combining the above-mentioned definitions of (weak) detectability and observability. We call the above property, from now on, *input/output to state stability (*IOSS*)*. The question, of course, is what interesting properties can be proved when using this definition. More specifically, one should ask if there are other different properties that could be called detectability.

Indeed, there is another, equally strong, contender for the role of detectability. This is the notion defined by requiring that there exist a "Lyapunov" function (positive for $x \neq 0$, radially unbounded) V defined on states, such that an estimate of the type

$$\dot{V}(x(t)) \ \le\ -\alpha_1(|x(t)|) + \alpha_2(|u(t)|) + \alpha_3(|y(t)|) \qquad (1)$$

holds along all trajectories (for some functions α_i of class $\mathcal{K}$ and α_1 of class $\mathcal{K}_\infty$, that is to say, unbounded). Notice the interpretation of this definition: the "energy" of states decreases when u and y are zero, and more generally when the state is large compared with the input/output data. In particular, consider the special case of (1) when there are no controls ($u \equiv 0$). For nonzero states, this says that $\dot{V}(x(t)) < \alpha_3(|y(t)|)$. In this restricted form, the definition has appeared frequently in the literature; sometimes

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*Here $|\xi|$ indicates Euclidean norm, $\left\|u|_{[0,t]}\right\|$ and $\left\|y|_{[0,t]}\right\|$ denote respectively the sup norms of the input and the output $y = h(x(t))$, where $x(t)$ is the solution with $x(0) = \xi$ and input $u(\cdot)$ on the interval $[0,t]$, the functions γ_i are of class $\mathcal{K}$, that is zero at zero, strictly increasing, and continuous, and β is a function of class $\mathcal{KL}$, i.e. it decreases to zero on t and is of class $\mathcal{K}$ on x.

instance [2]. It is easy to see that the "dissipation" characterization given by the estimate (1) implies IOSS (the proof is similar to the proof given below for the special case of $u \equiv 0$). **The main contribution of the current work is to investigate the converse implication**. In this paper, we show that, for systems with no controls, the IOSS property (in that case, we call it just OSS, output to state stability) implies the estimate. (We are fairly sure that this converse implication is true also for the general case when there are controls, but we were not able to write a complete proof in time for the conference deadline.) The precise statement is given in the next section, and the proof follows. The last section has a remark about i/o stabilization.

2. Output to State Stability

We next consider autonomous systems, i.e., systems with no inputs:

$$\dot{x} = f(x), \quad y = h(x), \tag{2}$$

where $f : \mathbb{X} \to \mathbb{X}$ is locally Lipschitz continuous and $h : \mathbb{X} \to \mathbb{R}^p$ is continuously differentiable, and where the state space $\mathbb{X} = \mathbb{R}^n$ for some n. We assume that $x = 0$ is an equilibrium, that is, $f(0) = 0$. We also assume that $h(0) = 0$. In what follows, we always use $x(t, \xi)$ to denote the trajectory of (2) with initial state ξ, and write $y(t,\ \xi) = h(x(t,\ \xi))$. This trajectory, and consequently also $y(t,\ \xi)$, is defined on some maximal interval $[0, t_{\max})$, where $t_{\max} = t_{\max}(\xi) \le +\infty$.

For systems without inputs, the IOSS property reduces to: there exists some $\beta \in \mathcal{KL}$, some $\gamma \in \mathcal{K}$ such that

$$|x(t,\ \xi)| \le \max\left\{\beta(|\xi|\,,\, t),\ \gamma\left(\left\|y|_{[0,t]}\right\|\right)\right\} \tag{3}$$

for all $t \in [0, t_{\max})$. If this property holds for system (2), then we say that the system is *output-to-state stable* (OSS).

Definition 2.1 An OSS-*Lyapunov function* for system (2) is any function V with the following properties:
(i) There exist $\mathcal{K}_\infty$-functions α_1 and α_2 such that

$$\alpha_1(|\xi|) \le V(\xi) \le \alpha_2(|\xi|), \qquad \forall\, \xi \in \mathbb{X}. \tag{4}$$

(ii) V is differentiable along trajectories, that is, for every trajectory $x(t,\ \xi)$ of (2), $V(x(t,\ \xi))$ is differentiable in t. Furthermore, there exist $\mathcal{K}_\infty$-functions α_3 and σ such that for every trajectory $x(t, \xi)$, and all $t \ge 0$,

$$\frac{d}{dt}V((x(t,\xi)) \le -\alpha_3(|x(t,\xi)|) + \sigma(|h(x(t,\xi))|)\,. \tag{5}$$

Our main result is as follows.

Theorem 1 *System (2) is* OSS *if and only it admits an* OSS-*Lyapunov function.*

We wish to note that the proof of the main result is *not* an immediate application of the converse theorem for the ISS property given in [8]. (It is tempting to think that it is enough to replace u by y in the ISS definition and hence obtain OSS, but the roles of controls and outputs are very different; most importantly, it is not possible to concatenate pieces of output trajectores and obtain a valid output, as it is the case with inputs, and this fact is essential in the proof in [8].) As a matter of fact, a proof along the lines of that in [8] would provide an infinitely differentiable function V of x. We do not yet know if this can always be insured.

Sufficiency is an easy consequence of this lemma, which is basically in page 441 of [6]) and is easy to prove:

Lemma 2.2 For each continuous and positive definite function $\alpha : [0,\ \infty) \to \mathbb{R}_{\ge 0}$, there exists a $\mathcal{KL}$-function β_α with the following property: for any absolutely continuous function $w : [0, T] \to R_{\ge 0}$ and any number $v^* \ge 0$, if, for all $t \in [0, T]$ it holds that

$$w(t) \ge v^* \implies \dot{w}(t) \le -\alpha(w(t)) \qquad \text{a.e. },$$

then $w(T) \le \max\{\beta_\alpha(w(0), T),\ v^*\}$.

To prove sufficiency, assume that the system (2) admits an OSS-Lyapunov function V satisfying (4)-(5). Let χ be the $\mathcal{K}_\infty$ function defined by $\chi(r) = \alpha_3^{-1}(2\sigma(r))$, and pick $\hat{\alpha}_3(s) = (1/2)\alpha_3 \circ \alpha_2^{-1}$. Finally, apply Lemma 2.2, to obtain a $\mathcal{KL}$-function $\beta_0 = \beta_{\hat{\alpha}_3}$. Now pick any initial state ξ and consider the trajectory $x(t) := x(t,\ \xi)$. We then have that:

$$|x(t)| \ge \chi(|h(x(t))|) \;\Rightarrow\; \frac{d}{dt}V(x(t)) \le -\hat{\alpha}_3(V(x(t)))\,.$$

It then follows that

$$V(x(t)) \ge \hat{\chi}(|h(x(t)|) \;\Rightarrow\; \frac{d}{dt}V(x(t)) \le -\hat{\alpha}_3(V(x(t))\,,$$

where $\hat{\chi}(s) = \alpha_2(\chi(s))$. Take any $T \in [0,\ t_{max})$, and consider $w(t) = V(x(t))$ and $v^* =$ the maximum of $\hat{\chi}(|h(x(t))|)$ restricted to $[0,\ T]$. The choice of β_0 then insures that

$$w(T) \le \max\left\{\beta_0(w(0),\ T),\ \hat{\chi}\left(\left\|y|_{[0,T]}\right\|\right)\right\}.$$

From this we obtain (3) with $\beta(s,r) = \beta_0(\alpha_2(s),\ r)$ and $\gamma(s) = \hat{\chi}(s)$.

The proof of the necessity part of Theorem 1 will be based on a sequence of preliminary technical results. From now on, we assume given an OSS system (2). Without loss of generality (take a larger β if necessary), we assume that $\beta(s,0) > s$ for all $s > 0$.

Lemma 2.3 There exists some $\mathcal{K}_\infty$-function ρ so that the following properties hold:

- For any $\xi \in \mathbb{X}$ and any $\tau \in [0, t_{\max})$, if $|x(t,\ \xi)| \ge \rho(|y(t,\ \xi)|)$ for all $0 \le t \le \tau$, then $\gamma(|y(t,\ \xi)|) \le |\xi|/2$, and hence $|x(t,\xi)| \le \beta(|\xi|\,,\ 0)$ for all $0 \le t \le \tau$. In particular, if $|x(t,\ \xi)| \ge \rho(|y(t,\ \xi)|)$ for all $t \in [0,\ t_{\max})$, then $t_{\max} = \infty$.

that for any $|\xi| \le r$ and any $T \ge T_{r,\varepsilon}$, if $|x(t,\ \xi)| \ge \rho(|y(t,\ \xi)|)$ for all $0 \le t \le T$, then $|x(t,\ \xi)| < \varepsilon$ for all $t \in [T_{r,\varepsilon},\ T]$.

Proof. Assume that system (2) is OSS. Without loss of generality, we may assume that γ in Equation (3) is of class $\mathcal{K}_\infty$.

Let $\alpha_0(s) = \beta(s,\ 0)$. Recall that we assumed that $\alpha_0(s) > s$ for all $s > 0$. Now let ρ be any $\mathcal{K}_\infty$-function satisfying the inequality $\rho(s) > \alpha_0(4\gamma(s))$ for all $s > 0$.

Pick any $\xi \ne 0$, and assume that $|x(t,\ \xi)| \ge \rho(|y(t,\ \xi)|)$ for all $0 \le t \le \tau$ for some $\tau \in (0,\ t_{\max})$. Then, at $t = 0$,

$$\gamma(|y(0,\ \xi)|)\ \le\ \gamma(\rho^{-1}(|\xi|)) \le \gamma(\rho^{-1}(\alpha_0(|\xi|))) < |\xi|/4.$$

Hence, $\gamma(|y(t,\ \xi)|) < |\xi|/4$ for all $t \in [0,\ \delta)$ for some $\delta > 0$. Let $t_1 = \inf\{t > 0 : \gamma(|y(t,\ \xi)|) \ge |\xi|/2\}$. Then $t_1 > 0$. Assume now that $t_1 \le \tau$. Then

$$\gamma(|y(t_1,\ \xi)|) = |\xi|/2, \quad \text{and} \quad \gamma(|y(t,\ \xi)|) < |\xi|/2,$$

for each $t \in [0,\ t_1)$, and hence for such t: $|x(t,\ \xi)| \le \alpha_0(|\xi|)$. Then, for each $0 \le t \le t_1$,

$$\gamma(|y(t,\ \xi)|) \le \gamma(\rho(|x(t,\ \xi)|) \le \gamma(\rho(\alpha_0(|\xi|))) < |\xi|/4.$$

By continuity, $\gamma(|y(t_1,\ \xi)|) \le |\xi|/4$, contradicting the definition of t_1. This shows that it is impossible to have $t_1 \le \tau$, and the proof of part 1 of the lemma is complete.

For each $r > 0$ and each $i = 1, 2, \ldots$, let $r_i := 2^{1-i}r$, and let T_r be any nonnegative number so that $\beta(r,t) < r/2$ for all $t \ge T_r$. Now, given any $r > 0$ and any $\varepsilon > 0$, let $k(\varepsilon)$ be any positive integer so that $2^{-k(\varepsilon)}r < \varepsilon$ and define $T_{r,\varepsilon}$ as $T_{r_1} + T_{r_2} + \ldots + T_{r_{k(\varepsilon)}}$.

Pick any trajectory $x(t,\xi)$ as in the statement of the lemma, defined on an interval of the form $[0,T]$, with $T \ge T_{r,\varepsilon}$, with initial condition $|\xi| \le r$, and satisfying $|x(t,\ \xi)| \ge \rho(|y(t,\ \xi)|)$ for all $t \in [0,T]$. Then, part 1 of the lemma implies that $\gamma(|y(t,\ \xi)|) < |\xi|/2$ for all such t. Therefore, for any $t > T_{r_1} = T_r$,

$$\begin{aligned} |x(t,\ \xi)| &\le \max\{\beta(|\xi|,\ t),\ |\xi|/2\} \\ &\le \max\{\beta(r,\ t),\ r/2\} \le r/2\,. \end{aligned}$$

Consider now the restriction of the trajectory to the interval $[T_{r_1}, T]$. This is the same as the trajectory that starts from the state $x(T_{r_1}, \xi)$, which has norm less than r_1, so by the same argument and the definition of T_{r_2} we have that $|x(t,\ \xi)| \le r/4$ for all $t \ge T_{r_2}$. Repeating on each interval $[T_{r_i}, T_{r_{i+1}}]$, we conclude that $|x(t,\ \xi)| < \varepsilon$ for all $T_{r,\varepsilon} \le t \le T$. ∎

Let ρ be any $\mathcal{K}_\infty$-function satisfying the conclusions of Lemma 2.3. Without loss of generality, one may assume that ρ is smooth on $(0,\ \infty)$. Consider the closed set:

$$\mathcal{D} = \{\xi \in \mathbb{X} :\ |\xi|\ \le \rho(|h(\xi)|)\}.$$

Observe that if $\mathcal{D} = \mathbb{X}$, then any smooth, proper, positive definite function V can be taken as an OSS-Lyapunov for each ξ, $\lambda_\xi = \inf\{t \in [0,\ t_{\max}) :\ x(t,\xi) \in \mathcal{D}\}$ (with the understanding that $\lambda_\xi = t_{\max}$ if $x(t,\xi) \notin \mathcal{D}$ for all t). Notice that $\mathcal{D}$ has the following two properties:

Property 1. There exists a $\mathcal{K}$-function ω such that for any $\xi \notin \mathcal{D}$, $|x(t,\ \xi)| \le \omega(|\xi|)$ for all $t \in [0,\ \lambda_\xi)$ (e.g., $\omega(r) = \beta(r,0)$).

Property 2. For any $\varepsilon > 0$, any $r > 0$, there exists some $T_{r,\varepsilon}$ such that for any $\xi \in \mathbb{X}$ with $|\xi| \le r$, if $\lambda_\xi > T_{r,\varepsilon}$, then $|x(t,\ \xi)| < \varepsilon$ for all $t \in [T_{r,\varepsilon},\ \lambda_\xi)$.

The function $\kappa_0(\xi) := |D(\rho \circ |h|)(\xi) f(\xi)|$ is well-defined and continuous on the set of ξ's such that $h(\xi) \ne 0$; thus we may pick a smooth function $\kappa : \mathbb{X} \to [0,\ \infty)$ such that $\kappa(\xi) \ge \kappa_0(\xi)$ whenever $|h(\xi)| \ge 1$.

It will be useful to introduce the auxiliary system which "slows down" the motions of the original system:

$$\dot{z}\ =\ \widehat{f}(z)\ =\ \frac{1}{1 + |f(z)|^2 + \kappa(z)} f(z), \quad y = h(z)\,. \quad (6)$$

To distinguish from the trajectories of (2), we use $z(s,\ \xi)$ to denote the trajectory of (6) at time s whose initial state is $z(0,\ \xi) = \xi$. Note that since the right-hand is bounded, solutions are well-defined for all times. We can relate the trajectories of (6) and (2) by the formula $z(s,\xi) = x(\tau(s),\xi)$, where $\tau(\cdot)$ is a suitable reparametrization of time. For this, it is only necessary to define, for each fixed trajectory $x(t) = x(t,\ \xi)$ defined on $[0, t_{\max})$, $t_{\max} = t_{\max}(\xi)$,

$$\sigma(t) := \int_0^t \Big(1 + |f(x(r))|^2 + \kappa(x(r))\Big)\ dr\ ,\ t \in [0, t_{\max})\,,$$

and $\tau := \sigma^{-1}$. It holds then that

$$z(s,\ \xi) = x(\tau(s),\ \xi), \qquad \forall\, s \in [0,\ \infty),$$

and

$$x(t,\ \xi) = z(\tau^{-1}(t),\ \xi), \qquad \forall\, t \in [0,\ t_{\max}).$$

So, in particular $z(s,\ \xi) \notin \mathcal{D}$ if and only if $x(\tau(s),\ \xi) \notin \mathcal{D}$. Let, for each $\xi \in \mathbb{X}$, $\theta_\xi = \sigma(\lambda_\xi)$ (and $\theta_\xi = \infty$ if $\lambda_\xi = t_{\max}$). Then $\theta_\xi = \inf\{s \ge 0 :\ z(s,\ \xi) \in \mathcal{D}\}$. *Claim:* The function $\theta(\xi) := \theta_\xi$ is lower semicontinuous.

Proof. Let $\xi_0 \in \mathbb{X}$. Pick a sequence $\{\xi_k\}$ that converges to ξ_0. Let $\theta_k = \theta_{\xi_k}$ and $\theta_0 = \theta_{\xi_0}$. We need to show that

$$\theta_0 \le \hat{\theta} = \liminf_{k \to \infty} \theta_k\ .$$

If $\hat{\theta} = \infty$, there is nothing to prove, so assume without loss of generality that $\lim_{k\to\infty} \theta_k = \hat{\theta} < \infty$. It follows by continuity that $z(\hat{\theta},\ \xi_0) \in \mathcal{D}$, and hence, $\theta_0 \le \hat{\theta}$, as required.

We now define $g : \mathbb{X} \to [0,\ \infty]$ by

$$g(\xi) = \sup\{|z(s,\xi)| :\ s \in [0,\ \theta_\xi)\} \quad (7)$$

for each $\xi \in \mathcal{E}$, where $\mathcal{E}$ is the open set:

$$\mathcal{E} = \mathbb{X} \setminus \mathcal{D} = \{\xi :\ |\xi| > \rho(|h(\xi)|)\}, \quad (8)$$

in s for all $s \in [0, \theta_\xi)$, if $\xi \in \mathcal{E}$. Also, according to Property 1 of the set $\mathcal{D}$, $g(\xi) \le \omega(|\xi|)$, $\forall\, \xi \in \mathbb{X}$, and hence,

$$|\xi| \le g(\xi) \le \omega(|\xi|), \quad \forall\, \xi \in \mathcal{E}.$$

Notice that, by Property 2, if $\theta_\xi = \infty$ then $z(s,\xi) \to 0$ as $s \to \infty$, and otherwise the sup in Equation (7) can be taken over the finite interval $[0, \theta_\xi]$; thus, for every ξ in $\mathcal{E}$ there is some $s_\xi \in [0,\ \theta_\xi]$ (or in $[0,\ \theta_\xi)$ if $\theta_\xi = \infty$) such that $g(\xi) = |z(s_\xi,\ \xi)|$. In particular, g takes only finite values.

Lemma 2.4 The function $g(\xi)$ is continuous at 0 and lower semicontinuous on $\mathcal{E}$. Furthermore, for any $\xi \in \mathcal{E}$, $g(z(s,\ \xi))$ is continuous (even locally Lipschitz) in s on $[0,\ \theta_\xi)$.

Proof. Pick $\xi \in \mathcal{E}$. For each $\varepsilon > 0$, by the continuity property of trajectories on initial states and times, there is some neighborhood $\mathcal{U} \subset \mathcal{E}$ of ξ, and some $0 < \delta < s_\xi$ such that for every $\eta \in \mathcal{U}$, $|z(s,\ \eta)| > |z(s_\xi,\ \xi)| - \varepsilon$ for all $s \in [s_\xi - \delta, s_\xi]$. By the lower semicontinuity of θ, one may assume that $s_\xi - \delta \le \theta_\xi - \delta < \theta_\eta$ for all $\eta \in \mathcal{U}$ (shrink $\mathcal{U}$ if necessary). It then follows that $g(\eta) \ge z(s_\xi - \delta, \eta) \ge z(s_\xi, \xi) - \varepsilon$, so: $g(\eta) > g(\xi) - \varepsilon$, $\quad \forall \eta \in \mathcal{U}$. This shows that g is lower semicontinuous.

The continuity of g at 0 follows from the facts that $g(0) = 0$ and that $0 \le g(\xi) \le \omega(|\xi|)$.

Since $|z(s,\xi)|$ is a locally Lipschitz function of $s \in [0, \theta_\xi)$, $g(z(s,\ \xi))$ also is (as for each $0 \le s \le t < \theta_\xi$ there is some $r \in [s,t]$ so that $0 \le g(z(s,\ \xi)) - g(z(t,\ \xi)) \le |z(r,\ \xi)| - |z(t,\ \xi)|$). ∎

Now define $V_0 : \mathbb{X} \to [0,\ \infty)$ by the formula:

$$V_0(\xi) = \int_0^{\theta_\xi} g(z(s,\ \xi))e^{-s}\, ds\,, \tag{9}$$

if $\xi \in \mathcal{E}$, and $V_0(\xi) = 0$ if $\xi \in \mathcal{D}$.

This integral is finite, because $g(z(s,\xi))$ is nonincreasing. Let $\xi \in \mathcal{E}$. If $\theta_\xi < \infty$, clearly from the definition, $V_0(\xi) < g(\xi)$, and if $\theta_\xi = \infty$ then the fact that $g(z(s,\xi)) \to 0$ as $s \to \infty$ (which follows from $z(s,\xi) \to 0$ as $s \to \infty$) insures that again in that case $V_0(\xi) < g(\xi)$.

Lemma 2.5 The function V_0 is continuous at 0 and lower semicontinuous on $\mathcal{E}$. Furthermore, for each $\xi \in \mathcal{E}$, $V_0(z(s,\ \xi))$ is differentiable in s on the interval $[0,\ \theta_\xi)$, and there one has

$$\frac{d}{ds}V_0(z(s_0,\ \xi)) \;=\; V_0(z(s_0,\xi)) - g(z(s_0,\ \xi)). \tag{10}$$

Proof. Let $\xi_0 \in \mathcal{E}$. Pick any $\varepsilon > 0$. Choose $0 < K < \theta_0 = \theta_{\xi_0} \le \infty$ so that $e^{-K} - e^{-\theta_0} < \varepsilon$. By the lower semicontinuity of θ_ξ, we know that there exists a neighborhood $\mathcal{U} \subset \mathcal{E}$ such that for every $\xi \in \mathcal{U}$, $\theta_\xi > K$, and hence, for every $\xi \in \mathcal{U}$,

$$V_0(\xi) = \int_0^{\theta_\xi} g(z(s,\ \xi))e^{-s}\, ds \ge \int_0^{K} g(z(s,\ \xi))e^{-s}\, ds. \tag{11}$$

ξ_0. Thus $z(s,\xi_k) \in \mathcal{E}$ for all $s \in [0,K]$, so by Fatou's Lemma and lower semicontinuity of g on $\mathcal{E}$,

$$\begin{aligned}
&\liminf_{k\to\infty} \int_0^K g(z(s,\xi_k))e^{-s}\, ds \\
\ge\ & \int_0^K \liminf_{k\to\infty} g(z(s,\xi_k))e^{-s}\, ds \\
\ge\ & \int_0^K g(z(s,\ \xi_0))e^{-s}\, ds \\
=\ & V_0(\xi_0) - \int_K^{\theta_0} g(z(s,\ \xi_0))e^{-s}\, ds \\
\ge\ & V_0(\xi_0) - (e^{-K} - e^{-\theta_0})\omega(|\xi_0|) \ge V_0(\xi_0) - \varepsilon\omega(|\xi_0|)\,.
\end{aligned}$$

Combining this with Inequality (11), one concludes that $\liminf_{k\to\infty} V_0(\xi_k) \ge V_0(\xi_0) - \varepsilon\omega(|\xi_0|)$ for this $\varepsilon > 0$. Taking $\varepsilon \to 0$ shows lower semicontinuity of V_0.

Continuity of V_0 at 0 follows from the continuity of g at 0 and the facts that $0 \le V_0(\xi) \le g(\xi)$ and $g(0) = 0$.

Next we establish differentiability along trajectories while in $\mathcal{E}$. We show that for every $\xi \in \mathbb{X}$, the function $V_0(z(s,\xi))$ is differentiable at s_0 if $z(s_0,\ \xi) \in \mathcal{E}$. Note that $z(s_1, z(s_2,\xi)) = z(s_1 + s_2, \xi)$. For any s_0 for which $z(s_0,\xi) \in \mathcal{E}$, we let $\eta = z(s_0,\xi)$. Then $z(s_0 + h, \xi) = z(h,\eta)$, and $\theta_{z(h,\ \eta)} = \theta - h$, where $\theta = \theta_\eta$, for all h in some neighborhood of zero. Thus,

$$\begin{aligned}
V_0(z(s_0+h,\xi)) &= \int_0^{\theta-h} g(z(s,z(h,\eta)))e^{-s}ds \\
&= e^h \int_h^{\theta} g(z(s,z(h,\eta)))e^{-s}ds
\end{aligned}$$

for all h small, from which it follows by continuity of $g(z(s,z(h,\eta)))$ on s that

$$\frac{d}{ds}V_0(z(s_0,\ \xi)) = V_0(\eta) - g(\eta) = V_0(z(s_0,\xi)) - g(z(s_0,\ \xi))\,,$$

and Equation (10) indeed holds. ∎

From $g(\xi) > V_0(\xi)$ for $\xi \in \mathcal{E}$, $\frac{d}{ds}V_0(z(s_0,\xi)) < 0$ for all s_0 such that $z(s_0,\ \xi) \in \mathcal{E}$. Moreover, this expression can be bounded away from zero on compacts:

Lemma 2.6 For each compact subset $\mathcal{K}$ of $\mathcal{E}$, $\inf_{\xi\in\mathcal{K}} (g(\xi) - V_0(\xi)) > 0$.

Proof. Pick a compact subset $\mathcal{K}$ of $\mathcal{E}$. Since $\mathcal{K}$ is compact, by lower semicontinuity of V_0, one knows that there is some $r_0 > 0$ such that $V_0(\xi) > r_0$, and so also $g(\xi) > r_0$, for all $\xi \in \mathcal{K}$. By Property 2, since $\mathcal{K}$ is bounded, there exists some $s_\mathcal{K} > 0$ such that if $\theta_\xi > s_\mathcal{K}$, then

$$|z(s,\ \xi)| < r_0/2 \le g(\xi)/2, \quad \forall\, s \in [s_\mathcal{K},\ \theta_\xi).$$

This then implies that

$$V_0(\xi) \;=\; \int_0^{\theta_\xi} g(z(s,\xi))e^{-s}\, ds$$

$$
\begin{aligned}
&\le \int_0 g(z(s,\xi))e^{-s}\,ds + \frac{g(\xi)}{2}\int_{s_\mathcal{K}} e^{-s}\,ds \\
&\le g(\xi)(1-e^{-s_\mathcal{K}}) + \frac{g(\xi)}{2}\,e^{s_\mathcal{K}} \\
&= g(\xi) - \frac{g(\xi)}{2}\,e^{-s_\mathcal{K}} \le g(\xi) - \frac{r_0}{2}e^{-s_\mathcal{K}}.
\end{aligned}
$$

Thus $g(\xi) - V_0(\xi) > \frac{r_0}{2}e^{-s_\mathcal{K}}$ on $\mathcal{K}$. ∎

We now let $\mathcal{E}_1 = \{\xi \in \mathbb{X} : \ |\xi| \ge 2\rho(|h(\xi)|)\} \subset \mathcal{E} \cup \{0\}$.

Lemma 2.7 There are real numbers $K, c > 0$ such that $V_0(\xi) \ge c\,|\xi|$ for all $\xi \in \mathcal{E}_1$ so that $|\xi| \ge K$. (This says, in particular, that the restriction $V_0|_{\mathcal{E}_1}$ of V_0 to $\mathcal{E}_1$ is proper.)

Proof. We will prove the lemma by first showing that if $\xi \in \mathcal{E}_1$ with $|\xi| \ge K_0$, where $K_0 = \rho(1) + 4$, then $\theta_\xi \ge 1$.

For this purpose, fix any $\xi \in \mathcal{E}_1$ with $|\xi| \ge K_0$. Assume $\theta := \theta_\xi \le 1$. Let $\eta = z(\theta,\ \xi)$. Since $\left|\widehat{f}(\zeta)\right| \le 1$ for all $\zeta \in \mathbb{X}$, it holds that $|\eta| \ge |\xi| - \theta \ge K_0 - 1$. By the definitions of η and θ, one has

$$
\rho(|h(\eta)|) = |\eta| \ge K_0 - 1, \tag{12}
$$

so also $|h(\eta)| \ge \rho^{-1}(K_0 - 1) > 1$. Thus, $|h(z(s,\eta)| > 1$ for all s near zero.

Claim: $|h(z(s,\ \eta))| > 1$ for all $s \in [-1,\ 0]$.

Assume the claim is false. Then there must exist some $-1 \le s_0 < 0$ so that $s_0 = \max\{s \le 0 : \ |h(z(s,\ \eta))| \le 1\}$. We have that for each $s \in (s_0,\ 0]$, $|h(z(s,\ \eta))| > 1$. Recall that $|D(\rho \circ |h|)(\zeta) f(\zeta)| \le 1$ for all ζ with $|h(\zeta)| \ge 1$. Thus

$$
\left|\frac{d}{ds}\rho(|h(z(s,\ \eta))|)\right| \le 1, \quad \forall\, s \in (s_0,\ 0].
$$

This, in turn, implies that

$$
\rho(|h(z(s_0,\ \eta))|) \ge \rho(|h(\eta)|) + s_0 \ge K_0 + s_0 - 1 > \rho(1),
$$

contradicting the definition of s_0. This proves the claim.

It follows from the claim that $|h(z(s,\ \xi))| > 1$ for all $s \in [0, \theta]$. Thus,

$$
\begin{aligned}
\rho(|h(\eta)|) &= |\eta| \ \ge\ |\xi| - \theta \\
&\ge 2\rho(|\xi|) - \theta \ge 2\rho(|h(\eta)|) - 3\theta
\end{aligned}
$$

from which it follows that $\rho(|h(\eta)|) \le 3\theta$, so from (12) we know that $K_0 \le 3\theta + 1 \le 4$, contradicting the choice of K_0. This shows that it is impossible to have $\theta \le 1$.

To complete the proof of the lemma, observe that if $\xi \in \mathcal{E}_1$ with $|\xi| \ge K_0$, then

$$
\begin{aligned}
V_0(\xi) &\ge \int_0^1 g(z(s,\ \xi))e^{-s}\,ds \\
&\ge g(z(1,\ \xi))(1-e^{-1}) \ \ge\ (|\xi| - 1)(1 - e^{-1})
\end{aligned}
$$

Since $|\xi| \ge K_0 > 2$, $|\xi| - 1 \ge |\xi|/2$. Hence, if $\xi \in \mathcal{E}_1$ with $|\xi| \ge K_0$, then $V_0(\xi) \ge c\,|\xi|$, where $c = (1-e^{-1})/2 > 0$. ∎

By Lemma 2.6, we know that

$$
\inf\{g(\xi) - V_0(\xi)\,,\ a \le |\xi| \le b, \xi \in \mathcal{E}\} \ >\ 0
$$

positive definite function α_3 such that

$$
g(\xi) - V_0(\xi) > \alpha_3(|\xi|)\,, \quad \forall\, \xi \ne 0,\ \xi \in \mathcal{E}\,.
$$

In particular, $\frac{d}{ds}V_0(z(s,\xi)) \le -\alpha_4(V_0(z(s,\xi)))$ for all $s \in [0,\ \hat{\theta}_\xi)$, where $\hat{\theta}_\xi = \inf\{s \ge 0 : \ z(s,\ \xi) \notin \mathcal{E}_1\}$, and $\alpha_4(r) = \alpha_3(\omega^{-1}(r))$ (using that $\omega(|\xi|) > V_0(\xi)$ on $\mathcal{E}$, and hence in particular on $\mathcal{E}_1$). Consequently, in terms of the original time parameter and trajectory:

$$
\frac{d}{dt}V_0(x(t,\ \xi)) \le -\alpha_4(V_0(x(t,\ \xi)))
$$

for all $t \in [0,\ \hat{\lambda}_\xi)$, where $\hat{\lambda}_\xi = \inf\{t \in [0, t_{\max}) : \ x(t,\ \xi) \notin \mathcal{E}_1\}$.

To construct a Lyapunov function defined everywhere, consider a smooth function $\varphi : \mathbb{X} \setminus \{0\} \to [0,1]$ with the property that

$$
\varphi(\xi) = \begin{cases} 1, & \text{if } \rho(|h(\xi)|) \le |\xi|/3 \\ 0, & \text{if } \rho(|h(\xi)|) \ge |\xi|/2 \end{cases}
$$

and φ is nonzero elsewhere. Note that such a smooth function exists because the two sets $\{\rho(|h(\xi)|) \le |\xi|/3\}$ and $\{\rho(|h(\xi)|) \ge |\xi|/2\}$ are two disjoint closed subsets of $\mathbb{X} \setminus \{0\}$. Define $\varphi(0) = 0$. Let, for each $\xi \in \mathbb{X}$,

$$
V_1(\xi) = \varphi(\xi)V_0(\xi) + (1 - \varphi(\xi))\,|\xi|^2\,.
$$

Consider any $\xi \in \mathbb{X}$ with $|\xi| \ge \max\{K, c\}$, where K and c have been chosen as in Lemma 2.7. If $\xi \in \mathcal{E}_1$ then the Lemma assures that

$$
V_1(\xi) \ge c\,|\xi| \tag{13}
$$

and if $\xi \notin \mathcal{E}_1$ then $V_1(\xi) = |\xi|^2$ so this inequality is also satisfied; thus (13) holds globally; in particular, V_1 is proper. Note also that V_1 is positive definite, since on $\mathcal{E}_1 \setminus \{0\}$ the function V_0 is nonzero (it is nonzero on all of $\mathcal{E}$) and on the complement of this set it is equal to $|\xi|^2$.

We summarize our conclusions as follows:

- V_1 is lower semicontinuous everywhere, and continuous at 0 with $V_1(0) = 0$.

- V_1 is positive definite and proper, and hence (being lower semicontinuous), there exists $\alpha_1 \in \mathcal{K}_\infty$ such that $V_1(\xi) \ge \alpha_1(|\xi|)$ for all $\xi \in \mathbb{X}$. Moreover, since $V_0(\xi) \le g(\xi) \le \omega(|\xi|)$, it follows that there exists some $\alpha_2 \in \mathcal{K}_\infty$ such that $V_1(\xi) \le \alpha_2(|\xi|)$.

- For any $\xi \in \mathbb{X}$, $V_1(x(t,\ \xi))$ is differentiable in t for all $t \ge 0$, and $\frac{d}{dt}V_1(x(t,\ \xi)) \le -\alpha_4(V_1(x(t,\xi)))$ if $|x(t,\xi)| \ge 3\rho(|h(x(t,\ \xi))|$ and there is some function ψ of class $\mathcal{K}$ such that

$$
\frac{d}{dt}V_1(x(t,\ \xi)) \le \psi(x(t,\ \xi)) \tag{14}
$$

whenever $\rho(|h(x(t,\ \xi))| < |x(t,\xi)| \le 3\rho(|h(x(t,\ \xi))|$.

$\frac{d}{dt}V_0(x(t,\ \xi)) \leq V_0(x(t,\ \xi)) \leq \omega(|x(t,\xi)|)$ everywhere on $\mathcal{E}$, and both φ and $|\xi|^2$ are smooth.

Finally, let $\pi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be any $\mathcal{K}_\infty$-function which can be extended as a C^1 function to a neighborhood of $[0,\ \infty)$ and such that $\alpha_5(r) := \pi(\alpha_1(r))\alpha_4(\alpha_1(r))$ is a $\mathcal{K}_\infty$ function. Define: $V_2(\xi) = \Phi(V_1(\xi))$, where $\Phi(r) = \int_0^r \pi(s)\,ds$. Then the following properties hold for V_2: (1) V_2 is lower semicontinuous, and

$$\hat{\alpha}_1(|\xi|) \leq V_2(\xi) \leq \hat{\alpha}_2(|\xi|), \tag{15}$$

where $\hat{\alpha}_i(r) = \Phi(\alpha_i(r))$, $i = 1, 2$; (2) V_2 is differentiable along trajectories, and

$$\begin{aligned} \frac{d}{dt}V_2(x(t,\ \xi)) &= \pi(V_1(x(t,\ \xi)))\frac{d}{dt}V_1(x(t,\ \xi)) \\ &\leq -\alpha_5(|x(t,\xi)|) \end{aligned} \tag{16}$$

whenever $|x(t,\xi)| \geq 3\rho(|h(x(t,\ \xi))|$.

Since $V_1(\xi) = |\xi|^2$ is smooth on the set $\mathcal{E}_3 := \{\xi :\ |\xi| \leq 2\rho(|h(\xi)|)\}$, it follows that if $|x(t,\xi)| \in \mathcal{E}_3$, then

$$\frac{d}{dt}V_2(x(t,\xi)) \leq \sigma_0(|x(t,\xi)|) \tag{17}$$

for some $\mathcal{K}_\infty$-function σ_0. Because of (14), one knows that if $2\rho(|h(x(t,\xi))|)\} \leq |x(t,\ \xi)| \leq 3\rho(|h(x(t,\xi))|)\}$, then

$$\frac{d}{dt}V_2(x(t,\xi)) \leq \sigma_1(|x(t,\ \xi)|) \tag{18}$$

for some $\sigma_1 \in \mathcal{K}_\infty$. Let

$$\sigma(r) = \max\{\sigma_1(3\rho(r)),\ \sigma_0(2\rho(r)) + \alpha_5(3\rho(r)).$$

Then, combining (16), (17) and (18), we get

$$\frac{d}{dt}V_2(x(t,\ \xi)) \leq -\alpha_5(|x(t,\ \xi)|) + \sigma(|h(x(t,\ \xi))|), \quad \forall t \geq 0. \tag{19}$$

This shows that V_2 is an OSS-Lyapunov function for the system. ∎

Remark 2.8 To find an explicit way of constructing the function π, we refer the reader to [3]. By Lemmas 11 and 12 in [3], one can construct a $\mathcal{K}_\infty$-function π which can be extended as a C^1 function to $[-1, \infty)$ such that $\pi'(r) \geq \frac{\pi(r)}{\alpha_4(r)}$, $\quad \forall r > 0$. With such a choice of π, inequality (16) becomes:

$$\begin{aligned} \frac{d}{dt}V_2(x(t,\ \xi)) &= \pi'(V_1(x(t,\ \xi)))\frac{d}{dt}V_1(x(t,\ \xi)) \\ &\leq -V_2(x(t,\ \xi)) \end{aligned} \tag{20}$$

whenever $|x(t,\xi)| \geq 3\rho(|h(x(t,\ \xi))|$. Consequently, the estimation (19) becomes

$$\frac{d}{dt}V_2(x(t,\ \xi)) \leq -V_2(|x(t,\ \xi)|) + \sigma(|h(x(t,\ \xi))|), \quad \forall t \geq 0, \tag{21}$$

for some $\sigma \in \mathcal{K}_\infty$. Hence, we obtain with the same proof the following sharper version of the theorem: an OSS-Lyapunov function V satisfying an estimation of type (21). □

3. A Remark on I/O Stabilization

Reasons of space preclude discussing here the relationship between the IOSS property and i/o stabilization. We merely point out the following fact: if there is for each state ξ an open loop control $u_\xi(\cdot)$ that drives ξ asymptotically to zero (and itself converges to zero), if the right-hand side $f(x,u)$ is an analytic function of (x,u), and if the system is IOSS, then there is a controller which, based only upon output information, produces a control that drives all states to zero. The procedure is roughly as follows: 1: apply a "random" input on some interval $[0,T]$; 2: pick any state $\xi = x(T)$ consistent with the observed i/o data (we assume here that there are no finite explosion times, so this data is indeed well-defined); and finally 3: pick a control u_ξ so that $x(t;\xi,u_\xi) \to 0$ and $u(t) \to 0$. We claim that this procedure indeed drives the internal state asymptotically to zero. Indeed, we know from the Universal-Input Theorem for analytic systems that a "generic" control will identify the final state $x(T)$ up to indistinguishability, that is, when we apply the further control u_ξ, the true state ζ at time T will produce the same output, going to zero, as ξ does. Now the IOSS property implies that the trajectory starting from ζ must also converge to zero. The full version of this paper will make this argument more rigorous, including a version in which input and output disturbances are taken into account.

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