Nonlinear Regulation: The Piecewise Linear Approach

EDUARDO D. SONTAG

Abstract—This paper approaches nonlinear control problems through the use of (discrete-time) *piecewise linear* systems. These are systems whose next-state and output maps are both described by PL maps, i.e., by maps which are affine on each of the components of a finite polyhedral partition. Various results on state and output feedback, observers, and inverses, standard for linear systems, are proved for PL systems. Many of these results are then used in the study of more general (both discrete- and continuous-time) systems, using suitable approximations.

I. Introduction

THIS paper approaches nonlinear regulation through the study of *piecewise linear* (PL) systems, with two quite different sets of goals: the study of PL systems per se (as a generalization of linear systems), and the use of the tools and methods developed for PL systems in the control of other, more general, classes of systems. This latter aspect may be seen as constituting one systematic approach to numerical nonlinear control, based on piecewise linear approximations.

These are various reasons which suggest that it may be worthwhile to investigate PL systems: simplicity of implementation, theoretical analysis, and calculation. The former is due to the fact that digital controllers based on such systems can be built easily using "if $P(x)$ then $f(x)$ else…" programs, where P, f involve only affine combinations and "greater than" comparisons. This shifting of computational emphasis from arithmetic to logic is especially convenient when using microprocessors. Simplicity of analysis is in principle less obvious, but the recent development of the beginnings of a "PL algebra" in Sontag [12] indicates that this may indeed be the case. The remarks in this paper illustrate this simplicity on system theoretic grounds. The last claimed advantage, regarding off-line calculation, is yet to be explored in any serious detail. Preliminary investigations indicate that, at least for systems of reasonably small dimension, low complexity algorithms may be available for analysis and synthesis. Among other advantages, it should be noted that the use of PL systems permits introducing thresholds and other discontinuities in a natural way that is not available in other algebraic approaches to nonlinear system theory.

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The author is with the Department of Mathematics, Rutgers University, New Brunswick, NJ 08903.

In order to make this paper self-contained, we review briefly the relevant definitions and results from [12]. A *PL subset* of a finite-dimensional real vector space V is the union of a finite number of relatively open polyhedra, i.e., of sets defined by (finitely many) linear equations $f(x)=a$ and linear inequalities $f(x)>a$. A *PL set* is a PL subset of some V. A *PL relation* R: $X\to Y$ between PL sets is one whose graph is a PL set, and similarly for PL *maps*. Equivalently, the map f is PL if there is a covering of X by PL subsets X_i such that the restrictions $f|X_i$ are all affine (=linear+translation). A useful way of defining PL sets is the following. Let L be the first-order language consisting of constants r and unary function symbols $r(\cdot)$ for each real r, variables $x_1, x_2, \cdots$, binary function symbol $+$, and relation symbols $>$, $=$. Then we have the following.

Lemma 1.1 [*12, Lemma (2.6)*]: Every sentence in L defines a PL set. ##

When applying the above, one can include in L labels for sets and functions already known to be PL; sentences will still define PL sets. The study of "elementary" problems, those expressible by finitely many such sentences, is considerably simplified by (1.1), as will be seen in Section II. Another very useful tool is the existence of selection (choice) functions.

Selection Lemma 1.2 [*12, Theorem (2.11)*]: Let R: $X\to Y$ be a PL relation with domain Z. There is then a PL map s: $Z\to Y$ with $s(x)$ in $R(x)$ for all x.

Alternative ways of stating (1.2) are as a "global implicit function theorem," through sections of congruences, or as a "PL axiom of choice." Checking the truth of a sentence in L, or constructing a set defined by such a sentence (as those in Section II), can be carried out via three basic algorithms: one for projecting polyhedra on hyperplanes, another for checking feasibility of a linear program, and a standard Boolean table. These procedures appear to be as central as matrix multiplication is in the case of linear algebra. The construction of sections in (1.2) can be also accomplished via linear programming methods. Thus, all constructions are "effective" in a very concrete sense, and recent linear programming developments imply polynomial time decidability of many types of formulas. Various useful "closed-form" representations for PL maps have been introduced by Kang and Chua [8], motivated by the network theoretic applications developed by Chua [1]; the development of software for PL algebra will probably include the use of such representations. There are already a consider-

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able number of results relating to PL maps (see, for instance, Fujisawa and Kuh [3] and the references therein) which deal with algorithms for checking various specific algebraic properties. Our proofs in this paper are, however, of a theoretical nature; the study of implementations should, of course, be an important project in itself.

A fixed but arbitrary PL set U (the "input value set") will be used throughout. A special element "0" will be singled out in U, to correspond to the notion of "no input being applied." Up to a translation, there is no loss of generality in taking 0 to be the zero vector in a vector space of which U is a PL subset. When output behavior is of interest, Y will denote a fixed but arbitrary PL set ("output value set").

A *system* $S=(U, X, Y, p, q)$ (where U, X, Y, or q may be omitted if clear from the context or if irrelevant) is given by a pair of maps p, q from $X\times U$ into X and Y, respectively, for some set X ("state set"). One interprets S as defining the equations

$$x(t+1)=p(x(t),u(t))$$
$$y(t)=q(x(t),u(t)), \qquad t=0,1,\cdots \tag{1.3}$$

with the $x(t)$, $u(t)$, $y(t)$ representing states, inputs, and outputs at time t. A *state output* system has q independent of u; S is *autonomous* if p is also independent of u. In both cases one simply drops the corresponding $u(t)$ in displaying (1.3). A *PL system* is one for which X is a PL set and both p, q are PL maps. A convenient way of specifying PL systems is via algol-like "if-then-else" programs, and this will be done when appropriate. Note that the class of PL systems includes in particular finite automata and linear systems; the former appear when every set is finite (hence PL); any map among such sets is obviously PL. Various ("hierarchical") combinations of automata and linear systems are also PL: for example, a finite counter added to a set of linear systems, deciding "which system" to use depending on the value of the counter. Linear systems with (polyhedral) input and state constraints (and/or saturation effects) are also modeled by the theory. More general nonlinear systems can be (under weak hypotheses) globally approximated arbitrarily close by PL systems. Of course, this generality means that one cannot expect to obtain computationally trivial solutions to all control problems, even though the basic procedures developed below are not hard to understand. Efficient tools for the linear algebra and combinatorics involved in the (off-line) computations will have to be developed, and this should have high priority in future research in this area. Similarly, the study of approximations by PL systems is in itself basically open, and it is clear that deeper results will depend upon a systematic use of the powerful results developed for nonlinear systems during the past decade.

This paper should be seen as just a first step in the development of a PL system theory. In contrast to most previous approaches to the modeling of systems with discontinuities ("sliding mode systems," "PL networks," etc.), the emphasis here will be first on discrete-time systems. This different emphasis allows for the use of the algebraic tools in [12], since the iterated transition maps are still PL. (Instead, for continuous-time systems with PL system maps, the evolution map does not inherit the PL structure.) Since the number of "linear pieces," of course, increases with the iterations, various compactness and other finiteness assumptions are used at various points. The application to non-PL systems is via approximations and, for continuous time, the introduction of time sampling. For the latter the theory is, then, one of sampled systems, suitable more for digital control than for classical analog control. The connections with the work of Aiserman, Gantmacher, and others (e.g., see Minorski [10, chapters 6–10]) on continuous time PL systems are not yet totally clear, however. For a less systematic but very interesting discussion of various problems for discrete-time systems with nonsmooth transitions, see Vidal [16].

The first part will deal with bounded-time problems for PL systems per se, while the second deals with "asymptotic" properties for PL and other systems. The number of problems left open for further research is very large. Problems for PL systems of decoupling, disturbance rejection, and many others treated successfully by the linear theory are not studied here, although it is reasonable to expect that similar methods will apply to these. Even for the problems which are treated, many improvements in statements and proofs should be possible, and open problems are pointed out throughout.

II. Elementary PL Control Problems

This section deals with the control and observation of PL systems over *finite* time intervals. Besides being of interest in themselves, the understanding of the corresponding problems is helpful in the study of asymptotic properties. The general approach is to express the properties of interest in the first-order language L, and to apply tools from PL algebra to obtain easly the desired results. The latter say in essence that for most reasonable bounded-time problems for PL systems, if a solution exists abstractly, then there is also a solution which can be implemented using PL systems. Throughout this part, S will denote a fixed PL system (p,q). The extension of p to input sequences will be denoted by P. An arbitrary PL subset Z of the state space X will be also fixed, with all definitions made relative to Z.

A. State Feedback

All the properties in this section will be stated relative to a fixed equilibrium state x, i.e., an x for which $p(x,0)=0$; for simplicity x will be taken to be zero. Both of the definitions below describe, for fixed n, elementary first-order properties of Z, since they include quantification only over states and input values.

Definition 2.1: The subset Z is (n-step, globally) *controllable* (to zero) if for each state x in Z there is some input sequence w of length at most n for which $P(x,w)=0$.

Definition 2.2: The subset Z is (n-step, globally) *finitely stable* if the state 0 is an (n-step) attractor relative to Z, i.e., if $P(x,0^*)=0$ for all x in Z and for any null input sequence 0^* of length (at least) n.

Remark 2.3: For discrete-time linear systems (with unbounded inputs) it is well known that, with $Z=X$, (2.1) is equivalent to the property that for any state x there be an input sequence driving x to the origin, with no *a priori* bounds on the length of such sequences. In fact, one may, by Cayley–Hamilton, take n there as the dimension of the system. For more general systems such a property is, of course, not equivalent to the above (bounded-time) controllability—any nonnilpotent linear system with bounded controls serves as a counterexample. Consequences of such more general definitions will be studied in Section III. Clearly, various technical restrictions (of compactness, convexity, etc.) could in any case be used, as linearity, to ensure the above controllability under weaker conditions. ##

The analog of the following result is well known in the linear case ("dead-beat controllers").

Theorem 2.4 The subset Z is controllable if and only if there exists a (feedback) PL map K: $X\rightarrow U$ with $K(0)=0$ such that Z is finitely stable for the closed-loop system $x(t+1)=p(x(t), K(x(t)))$.

Proof: When a stabilizing feedback exists, the original set Z must be controllable [use $u(t)=K(x(t))$]. The converse is proved via a dynamic programming argument: let $X(t)$ denote the set of states controllable to zero in t (but not less) steps. Thus, $X(0)$ is just $\{0\}$ and the union of the disjoint sets $X(t)$, $t=0,\cdots,n$, covers Z for some n. Each of the $X(t)$ is a PL set, since it can be defined by first-order sentences. Consider for each $t=1,\cdots,n$ the set $R(t)$ consisting of all the pairs (x,u) with x in $X(t)$ and $p(x,u)$ in $X(t-1)$, seen as a PL relation with domain $X(t)$. By the selection lemma there is a PL section $K(t)$: $X(t)\rightarrow U$ of $R(t)$. Define $K(0)$ as the zero map and K as the union of the $K(t)$, and extend K to all of X arbitrarily. Any state x of Z is in some $X(t)$ and is hence driven to zero in t steps under $u(t)=K(x(t))$. (Note this K gives a minimal-time controller.) ##

As an easy illustration take the (almost-) linear system for which Z is the set of real numbers x with $|x|\leqslant B$, X is the union of Z and a distinguished state e, U is the set of reals with $|u|\leqslant 1$, and whose transitions are given by: $x(t+1)=x(t)+u(t)$ when $x(t)$ is not e and the right-hand side has absolute value less than B, and $x(t+1)=e$ otherwise. (This kind of system arises if B is a bound on the domain of linearity of a given model and e denotes an error situation.) Then every state except e is controllable, and a natural (and obvious) choice of K on Z is $K(x)$:=if $|x|>1$ then $-\operatorname{sign}(x)$ else $-x$. ##

B. Observers

The iterated output map h gives the output resulting from state x after the application of the input sequence $w=u(0)\cdots u(t)$, i.e., $h(x,w)$ is $q(P(x,u(0)\cdots u(t-1)),u(t))$; the sequence of all the outputs is $H(x,w)=h(x,u(0)),\cdots,h(x,w)$.

Definition 2.5: The input sequence w *determines* the states of Z iff the outputs $H(x,w)$ for x in Z are sufficient to uniquely determine $P(x,w)$.

Remark 2.6: A w as above is called a "homing sequence" in automata theory (e.g., see Gill [5]). For linear systems *any* input sequence whose length is at least equal to the dimension of the system serves to determine Z, provided that at least one such w exists, and in particular observable systems admit such w. For various classes of nonlinear systems there are similar results; see Sontag [11] and Sussmann [14]. ##

The *cascade* $S\#S'$ of two systems $S=(U,X,Y,p,q)$ and $S'=(U\times Y, X', Y', p', q')$ is the system $(U, X\times X', Y', p^*, q^*)$, where

$$p^*((x,x'),u)=\big(p(x,u),p'(x',(u,q(x,u)))\big), \tag{2.7}$$

$$q^*((x,x'),u)=q'(x',(u,q(x,u))). \tag{2.8}$$

Definition 2.9: The system S' is an *observer* for (the states of) Z, relative to a set W of input sequences, iff $Y'=X$ and for each initial state (x,x') of $S\#S'$ having x in Z, and each sequence w in W, $h^*((x,x'),w)=P(x,w)$.

If S' and W are as above then any w in W determines the states of Z. The converse is as follows.

Theorem 2.10: Let W be a set of length-n input sequences each of which determines the states of Z. There exists then a PL observer S' for Z relative to W.

Proof: Replace W by the set consisting of all those w of length n which determine Z. This (larger) set is PL, since (2.5) is a first-order definition. Let D be the PL subset of Y^n consisting of all $H(x,w)$ with x in Z and w in W. Consider the set of all triples $(H(x,w),w,P(x,w))$ with x in Z and w in W. This is the graph of a map f: $D\times u^n\rightarrow X$ and it is a PL set, so f is a PL map. The observer S' is now constructed as follows. The memory of S' is composed of a pair of shift registers which store the past $n-1$ input and output values, as well as of an extra register for storing the last output y' of S'. If the past $(n-1)$ outputs and inputs together with the present (y,u) constitute a pair in the domain of f, then S' outputs the corresponding value of f. Otherwise, S' simply evaluates $p(y',u)$, where u is the present input. Clearly, S' is an observer, and in fact the output of S' is equal to the state of S for *any* time $t\geqslant n$, as long as the first n input values constitute a sequence in W. ##

C. Input–Output Regulation

There are various ways to pose problems of input–output regulation, i.e., problems of constructing a system S' (regulator) which, on the basis of the *output* of a given system S, calculates the inputs necessary to force S to behave in a desired way. Only stabilization is considered; other topics like disturbance rejection, tracking, etc., will not be studied in this note. The situation here has, of course, strong overlaps with the results for linear systems (e.g., see Wonham [17] for the linear case), and with those for finite automata (for which see Gatto and Guardabassi [4]). Even when restricting attention to I/O stabilization there are still many possibilities in the choice of definitions. The main choice is in how one wants to model a situation in which unobservable (finite-time) disturbances may influence S. If S' is to regulate S but is "unaware" of *when* a disturbance occurs (except through its effect on the I/O behavior of S), then S' must regulate S even when S' starts at an arbitrary initial state (namely, the one in which S' ends up at the end of the action of the disturbance). If instead S' can be made aware of the disturbance, it could reset itself to a special state and only regulation when starting at that state is required. (One could also model disturbances as extra inputs.) Alternative ways of thinking about this are in terms of "time (in)variance" of the regulator, or in terms of the "degree of feedback" in the regulator construction. A related situation has been studied in automata theory by Gatto and Guardabassi, who introduced the terms "synchronous" and "asynchronous" regulation. Since output feedback will be involved, the system S will in this section be a state output system. As in Section I, there is a fixed equilibrium state 0.

Definition 2.11: The PL system $S'=(Y, X', U, p', q')$ is an (n-step) *weak regulator* for the subset Z of X iff there is a state x' in X' such that, for each x in Z, solving the closed-loop equations

$$x(t+1)=p(x(t), q'(x'(t), q(x(t)))) \quad (2.12a)$$

$$x'(t+1)=p'(x'(t), q(x(t))) \quad (2.12b)$$

with $x(0)=x$, $x'(0)=x'$ results in $x(t)=0$ for all $t\geqslant n$. A *strong regulator* is one for which this property holds for every x' in S'. A weakly or a strongly *regulable* system is an S which admits a regulator S' for $Z=X$.

Proposition 2.13: Assume that there exists an input sequence w which determines the states of Z, and that $P(Z\times\{w\})$ is controllable. There is then a weak regulator for Z.

Proof: Just cascade an observer with a system that computes a controlling input. That is, S' first feeds w into S, for the first r (=length of w) instants, storing corresponding outputs of S. Once that the state x of S is determined, S' sends the necessary input to control and sends zero thereafter. S' can be designed as a PL system through the application of the selection lemma to the relation $R(x):=\{v \text{ of length } n \mid P(x, v)=0\}$ (where $n=$ controlling time). ##

Theorem 2.14: Assume that there exists an integer r such that every sequence of length r determines the states of Z, and that $P(Z\times U^r)$ is controllable. Then there exists a strong regulator for Z.

Proof: Let K be a PL feedback map as in (2.4) and let S' be an observer for Z relative to W, where W denotes the set of all input sequences of length r, constructed as in (2.4). Define S'' using the same X' and p' as S' but with output $q''(x', q(x)):=K(q'(x', q(x)))$. For any $x(0)$, $x'(0)$, the sequence $w=q''(x'(0), q(x(0))),\cdots, q''(x'(r-1), q(x(r-1)))$ determines the states of Z, so the output $q'(x'(t), q(x(t)))$ is equal to $x(t)$ for any $t\geqslant r$. After $t=r$, therefore, $q''(x'(t), q(x(t)))=K(x(t), q(x(t)))$, so $x(t)$ is zero for all $t\geqslant r+n$. ##

Although the observability assumption used above is probably too strong in general, it may very well be that (for large enough r) it will apply to wide classes of systems [cf. (2.6)]. In any case, it can usually be obtained by restricting U or (equivalently) avoiding values of K that result in nondetermining inputs. Under suitable continuity assumptions, K can be perturbed while still controlling, as in the constructions in Section III.

D. System Inverses

The notion of inverse system is an input–output one, so one needs here a few more definitions. An *initialized* PL system (S, x) (or just S, if x is clear) is a PL system S with a choice of a state x in X. The *input–output map* of (S, x) is the map f which sends each input sequence w into the output sequence $H(x, w)$. When $U'=Y$, systems can be *composed in series* as follows: $S'':=S'\cdot S$ is the system whose state space is $X\times X'$ (in that order) and with $p''((x, x'), u)=(p(x, u), p'(x', q(x, u)))$ and $q''((x, x'), u)=q'(x', q(x, u))$. For initialized systems (S, x), (S', x') one defines $S'\cdot S$ as initialized at (x, x'). The I/O map $f'\cdot f$ of $S'\cdot S$ is then the composition of f and f'. A special type of I/O map of interest is the *delay* d^* associated with a nonnegative integer d:

$$d^*(u(0),\cdots, u(t-1)) = (0,\cdots,0, u(0), u(1),\cdots, u(t-d-1)) \quad (2.15)$$

(input preceded by d zeros). Note that d^* can be realized by an obvious PL system (with $U=Y$). A (left) *d-invertible* f is one for which there exists some f' with $f'\cdot f=d^*$. It is of interest in coding and other applications to characterize those systems whose I/O map is d-invertible.

Theorem 2.16: The I/O map f of (S, x_o) is d-invertible if and only if, for each state x reachable from x_o, and for each pair of input sequences $w=u(0)\cdots u(d)$ and $w'=u'(0)\cdots u'(d)$, the equality $H(x, w)=H(x, w')$ implies that $u(0)=u'(0)$.

Proof: Assume that f is d-invertible, $f'\cdot f=d^*$, f' the I/O map of S'. Let $x=P(x_0, w^*)$, with w^* of length r. If $H(x, w)=H(x, w')$, the outputs $f(ww^*)$ and $f(w'w^*)$ are equal and have length $r+d+1$. The output of $S'\cdot S$ at time $r+d+1$ is therefore equal for both inputs. But $f'\cdot f=d^*$ means that the output at time $r+d+1$ is $u(0)=u'(0)$, as wanted. Assume conversely that the required property holds for H. Let M: $X\times Y^{d+1}\to U$ be a PL map having the following properties: if there exists an input sequence $u(0)\cdots u(d)$ such that $H(x, u(0)\cdots u(d))=y(0)\cdots y(d)$, then $M(x, y(0)\cdots y(d))$ is any $u(0)$ obtained in this way. (Note that $u(0)$ may not be unique, since the property on H holds only for reachable states, but the latter cannot be singled out in the definition because they may not constitute a PL subset of X.) If $(x, y(\cdot))$ is such that no such $u(\cdot)$ exists, $M(x, y(\cdot))$ is zero. By the selection lemma and first-order definition methods, there exists such an M. The system S' is now defined. Its state space is the product of d copies of Y, X, and $[d]$, where the latter denotes the set of integers $\{0,\cdots,d\}$, thought of as a counter. The initial state of S' will have this counter at zero, all copies of Y at zero, and X at the initial state of S. The dynamics of S' are as follows: the d copies of Y will constitute a shift register storing the past d values of the input to S'. While the value

i of $[d]$ is less than d, the states of the copy of X will remain unchanged and the value of i increases by 1 at each step. When i reaches d, it remains there, and states in X are updated according to the original transition function p and using the inputs $M(x, y)$ where the first d terms of y are the values stored in the shift register (last d inputs) and the last entry of y is the current input to S' (output of S). The output of S' is given by: zero if $i<d$, and $M(x, y)$, with y as above, if $i=d$. ##

Note that the existence of an abstract (non-PL) d-inverse S' of S would already imply the above property. Thus, invertibility implies PL invertibility. The only nontrivial part in the above proof is the fact that there is a PL map M with the desired properties. The construction of S' is just a way of expressing the computations involved in d-inverting. This kind of approach is very much the general one with elementary properties of PL systems: after a certain map or set is shown to be PL, the "system" implementing the solution is just the obvious algorithm: PL systems are general enough to accommodate the needed control structures. Related to the results in this section, we leave as a suggestion the study of problems of feedback equivalence for PL systems using the approach of Hautus [6] together with the methods here.

III. Asymptotic Behavior

Since PL systems are defined in a finitary way (a PL map can only have finitely many affine components), the study of asymptotic problems for such systems is more delicate than that in Section II. Infinite-time problems are of interest because they often more accurately describe the behavior of real systems, and also because they transfer more easily under approximations of a non-PL system by a PL system and vice versa. The general approach in this section is to impose appropriate compactness conditions and to couple this with weak local assumptions. Again the emphasis is in proving that abstract existence of the various types of controllers implies the existence of PL controllers; checking the first of these (for non-PL systems) is itself a major (and active) area of system theory, in whose study the methods here may be useful numerically.

A. Asymptotic Control: Discrete Time

First, PL systems are treated. The following assumptions will hold unless otherwise stated: X and U are *connected* PL sets, p is *continuous*, and Z is *compact*. (These hypotheses are much stronger than those actually needed in the different proofs, but they appear to be natural enough to be used as a blanket assumption.) Statements about open or closed sets are always to be understood with respect to the topology of X as a PL subset of a given Euclidean space. The following definition will be used for both PL and non-PL systems, and for Z not necessarily compact.

Definition 3.1: The subset Z is *asymptotically controllable* (a.c.) (to zero) iff for each state x in Z there exists an infinite input sentence w such that the limit of $x(t)=P(x,w^t)$ exists and is zero. (Here w^t indicates the subsequence $w(0),\cdots,w(t)$.) The subset Z is (globally) *asymptotically stable* (a.s.) iff, for every x in Z, and with w constantly equal to zero,

(a) $x(t)$ is bounded if $x(0)=x$;

(b) for each neighborhood V of zero there exists a neighborhood W of x and a $T>0$ such that $z(t)$ is in V for any $z(0)$ in $W\cap Z$ and any $t\geq T$.

The above definition of a.s. is equivalent to the standard Lyapunov notion (e.g., see Lee and Markus [9]), but relativized to Z. When Z contains zero, the definition is equivalent to asymptotic stability of the origin plus all solutions starting at the subset Z converging to zero. In the results below, which already assume a local condition, only convergence to zero will be nontrivial to obtain. The same definition will be used later for continuous time systems. If a feedback law exists inducing closed-loop asymptotic stability for Z then Z is a.c. The interest here is in the converse statement, dealing essentially with the choice of infinite input sequences via a "finite" feedback. A first sufficient condition for this to be possible is the following.

Lemma 3.2: Let Z be asymptotically controllable. Assume that there is a PL neighborhood V' of zero and a continuous PL map L: $V'\to U$ such that, for some neighborhood V of 0, the iterates $q^i(x)$ are always in V' for any x in V, and V is a.s. for the system with transitions $q(x):=p(x, L(x))$. There exists then a feedback PL map K: $X\to U$ for which Z is a.s.

Proof: Without loss of generality, take V to be a connected open PL neighborhood of zero whose closure C is compact and satisfies the same property. For every x in C, there is some positive j such that the iterate $q^j(x)$ is back in V. Thus the (open) preimages $q^{-i}(V)$, $i\geq 1$, cover C, and by compactness these cover for $i=1,\cdots,j$, some j. Let D be the union of the sets V, $q(V),\cdots,q^j(V)$. Since p and L are PL maps, D is a PL set, and it is invariant under q (if x is in $q^j(V)$, write $x=q^j(z)$ for some z in V; then $q^i(z)$ is in V for some i between 1 and j, and hence $q(x)$ is in $q^{j-i+1}(V)$). Let $V(i)$ denote the set of elements of X which can be controlled into V in i steps. Each $V(i)$ is the union, ranging over all w of length i, of the preimages of V under the maps $P(\cdot,w)$. By continuity of P with respect to x, every $V(i)$ is open. Since each $V(i)$ is also the projection on X of the set consisting of all pairs (x,w) with w of length i and for which $P(x,w)$ is in V, these are also PL sets. By asymptotic controllability, the union of the $V(i)$ covers Z. By compactness of Z, there is some r so that $V(1),\cdots,V(r)$ already cover the set Z. Let $W(i)$ be the complement of D and $W(i-1)$ in $V(i)$ (where $W(0)=D$). Let K be equal to L on D, and equal to a selection function on the $W(i)$, $i=1,\cdots,r$, in such a way that $K(x)$ is an input sending states of $W(i)$ into $V(i-1)$. Outside the $W(i)$, K is arbitrary. Any state x in Z is in some $W(i)$, so it will be sent in at most r steps into D; after that the feedback L takes over, driving $x(t)$ towards 0. Thus, states converge to zero, and the proof follows from the fact that the origin is already a.s. for the local system. ##

Somewhat less than asymptotic controllability of Z is needed: it is enough to have "approximate controllability," i.e., existence of finite sequences driving the states of Z arbitrarily close to zero. The lemma could be extended to include the case in which it is required that Z also be invariant under the closed-loop transitions. A feedback K giving such an invariance will exist iff asymptotic control can be assumed without the sequence $x(\cdot)$ ever leaving Z. The proof is basically the same, only the $V(i)$ should be now defined as the set of elements of the subset Z which satisfy the necessary properties. Another extension would be to prove that controls outside D may in fact be chosen piecewise constant. This is not difficult to prove using compactness, but is relatively less interesting, due to the fact that the number of "pieces" and/or control times now become large in almost any instance (for a trivial example, consider an n-dimensional linear system $x(t+1)=x(t)+u(t)$, with U also n-dimensional, and with V' a small neighborhood of zero and Z a large set. The obvious feedback $u:=-x$ is linear (and so PL), but there is no piecewise constant feedback which uses "few" pieces).

The hypothesis of (3.2) holds, for example, if it is known that some controllability set $X(t)$ contains the origin in its interior (then L is as in Section II-A). A more interesting case is that in which p is linear at the origin, i.e., when there is a neighborhood of $x=0$, $u=0$, where p is defined and linear, and where this linear part is itself asymptotically controllable. In other words, $p(x,u)=Fx+Gu$ for small x, u, and the modes of F corresponding to eigenvalues of at least magnitude one give a reachable system. In the linear neighborhood a suitable feedback exists by the linear theory. Note that such a condition on (F,G) is "generic" in any reasonable sense. This argument generalizes to the non-PL case. The basic method there will be: for states near the origin a linear controller is used; for other states, a controller designed using a PL approximant will be sufficient. Discrete time is treated first.

Theorem 3.3: Let S be a discrete-time system with transitions $x(t+1)=p(x(t),u(t))$, whose state space X and input value set U are PL, connected, and contain 0 in their interiors, and with p continuous and p differentiable in a neighborhood of $x=0$, $u=0$. Let Z be a compact subset of X. Assume that Z and the linearization (F,G) of S at the equilibrium state 0 (i.e., $F=(\partial p/\partial x)(0,0)$, $G=(\partial p/\partial u)(0,0)$) are both a.c. There exists, then, a feedback map $K: X \rightarrow U$ such that Z is a.s. for the corresponding closed-loop system.

Proof: Let K be a linear map such that all eigenvalues of $A:=F+GK$ have absolute value less than one. The notation $|x|$ for a vector will be used for the box norm max $\{|x_1|,\cdots,|x_n|\}$; $|A|$ is the corresponding operator norm. "Distance" will refer to this norm. Since powers of A converge to zero, there is some t with $|A^t|<1/4$. Let $q(x):=p(x,Kx)$, for those x in X for which Kx is in U. Pick a neighborhood V of 0 small enough so that, for all x in V, $q^i(x)$ is defined and differentiable for $i=0,\cdots,t$. Further, V should be small enough that the closure of the union of the images of these $q^i(V)$ is also inside the domain of differentiability of p. Since q^t is differentiable, one may write, for x in V, $q^t(x)=A^tx+h(x)$, with $h(x)$ of order $o(x)$. Taking a smaller V if necessary, one may assume that $|h(x)|<(1/4)|x|$ for all x in V. Further, V can be taken as a box $|x|<a$. It follows that, on V, $|q^t(x)|<(1/2)|x|$, and in particular V is invariant under the contraction q^t. Moreover, $\{q^i(x)\}$ converges to zero, for any x in V, since $q,\cdots,q^{t-1}$ are all continuous and the subsequence $\{q^{it}(x)\}$ converges to zero. Let C be the compact box consisting of those x with $|x|\leqslant(a/2)$. Since Z is a.c., there is for each x in Z an input sequence w with $P(x,w)$ in the interior $i(C)$ of C. Thus, the preimages of $i(C)$ under the continuous maps $P(\cdot,w)$ cover Z, and by compactness a finite subset of the w is enough to cover. Let the integer r and the compact PL subset U' be such that every x in Z can be controlled to $i(C)$ (and hence to C) in at most r steps using input values in U'. Replace U by U'. Consider the union of all sets $P(Z\times U^i)$, for $i=0,\cdots,r$. This is again compact and is hence contained in a compact PL subset X' of X.

Let D' be the union of the images $(q^i(V))$ for $i=0,\cdots,t-1$. If x is in D' it returns to V, in fact to the interior of C, in at most t steps. So D' is also q-invariant. Let D be an open PL neighborhood of D' with the property that all $q^i(D)$ are well defined and also that $q^i(D)$ is eventually inside V. There is always such a D, because, for each x in the (compact!) closure of D', $q^i(x)$ is in C for some $i<t$, and $q,\cdots,q^{t-1}$ are all continuous. If x is any state in D then applying the linear feedback $u:=Kx$ results in x being driven asymptotically to zero. The global feedback K will be constructed as an extension of the map K as defined on D. The global K, which will in fact be defined only on X' (but can be extended arbitrarily outside X'), will in particular drive any state in Z into D. After reaching D, the "local" K takes over. Since the local system is a.s., only convergence to zero needs to be verified.

Consider the set C' of those states in X' which can be driven into C using a single input in U. (This set is compact.) Let p^* be a PL map defined on the product of X' and U and such that $p^*(x,u)$ is uniformly at distance less than $(a/4)$ of $p(x,u)$ (e.g., a continuous p^* which is a suitable simplicial approximation of p, with values in the convex hull of X'—or, for a much cruder approximation, a piecewise constant p^*). For each x in C' there is then some u such that $p^*(x,u)$ is at distance less than $a/4$ from C. Let V' be the open set consisting of all x in X' which can be so controlled in one step via the PL map p^* into a state at a distance less than $a/4$ from C. Since the latter is a compact PL set, V' is also PL. By the selection lemma, there exists a PL map K' defined on V' and with values in U such that the distance of $p^*(x,K'(x))$ to C is always less than $a/4$. Thus, $p(x,K'(x))$ is in V for any x in V'. Extend K to V' using K' outside D. The argument may be repeated now using C', V' instead of C, V; instead of a as above, one uses now a number a such that the distance from C' to the complement of V' is $a/2$. The only difficulty is that C' is not necessarily PL. But since C' is a compact subset of the open set V', there is always a PL C^* lying between C' and V', so C' can be simply replaced by this (larger) C^* if necessary. Repeating, a new p^* is defined on a suitable V'', and a K'' results which extends the previous K to V''. By the definition of r, after r steps the resulting C' already covers Z. Extend K arbitrarily outside the last V'. Under $u=K(x)$, a state x in Z is driven in at most r steps into D, as required. ##

What should be emphasized of the above result is the underlying idea of constructing an approximating PL system S^* (for example, the successive p^* on the V' that appear in the proof) and building a PL controller for S^*; if S^* is sufficiently close to S, the same controller works for the latter. (Introducing a uniform topology for systems, one could make this precise: there is a neighborhood of S such that any PL system in this neighborhood, equal on V to the linearization of S at zero, is also a.c.) The method could be reversed to give a (sufficient) test for the controllability of the original S (based on checking controllability of PL systems); if S^* is controllable, and if S is sufficiently close to S^*, it is itself controllable. It would, of course be desirable to have a version of this stated directly in terms of S.

Example 3.4: As a simple illustration, consider the system with $X=U=\mathbb{R}$, Z the set of states with $|x|\leqslant 3$, and transitions $x(t+1)=x(t)^2+x(t)+u(t)$. In this very simple case there is, of course, an obvious non-PL controller, namely, $u:=-x^2-x$, but we wish to obtain a PL solution. For this note that the linear feedback $K(x):=-x$ stabilizes for $|x|<1$. Interpolating linearly on the complement, one obtains for an approximant S^*:

$$p^*(x,u)=-3x-3+u \text{ on } [-3,-1] \tag{3.5a}$$

$$p^*(x,u)=5x-3+u \text{ on } [1,3]. \tag{3.5b}$$

The obvious controller $K(x):=3x+3$ on $[-3,-1]$, $K(x):=3-5x$ on $[1,3]$ when applied to the original system already happens to yield stability. Indeed, $|p(x,K(x))|<1$ for all x except $x=2,\ -2$. But in those cases the next state is -1, which is driven to zero in the next iteration. (Of course, one can not expect that such a simple procedure will always work, and better approximations of S may be necessary.) ##

Generalizations of (3.3) could proceed in various directions. For instance, X could be taken to be a more general topological space. One could then expect in that case an analogous argument using triangulations of X, but the detailed work remains to be done. Similarly, the control space U could be taken as a more general space.

B. Continuous-Time State Feedback

Consider now continuous-time systems S with X, U, PL open connected sets, f continuous [this can be weakened: see (3.9)], and such that

$$(dx/dt)(t)=f(x(t),u(t)) \tag{3.6}$$

has a well-defined unique solution for $t\geqslant 0$ when starting at any $x(0)$ in X and given any $u(\cdot)$ piecewise continuous with values in U. (By definition such u's will be assumed to be bounded on finite time intervals.) The definition of asymptotic controllability is the obvious analog of the one before. The following result proves that sampled control is (almost) always possible using PL feedback.

Theorem 3.7: Let S be as in (3.6), with f differentiable in a neighborhood of zero. Assume that the linearization (A,B) of f at 0 gives an a.c. linear system and that the compact subset Z of X is a.c. in the system S. Then there exist a PL map K: $X'\to U$ and a positive real number d, such that, for the system (3.6) with the feedback

$$u(t):=K(x(id)) \quad \text{if } id\leqslant t<(i+1)d, \tag{3.8}$$

the set Z is asymptotically stable.

Proof: By an easy modification of the argument in Kalman *et al.* [7, Theorem 12] one shows that there exists an integer j such that the sampled linear system (F,G) obtained from (A,B) using a sampling period 2^{-j} is a.c. as a discrete-time system. Let S' be the discrete-time system with the same X and U as S and with $p(x,u):=$ solution of (3.9) at time 2^{-j} for $x(0)=x$ and $u(t)=u$ (constant). (One says that S' is obtained through *sampling* of S at rate 2^j.) Let L be such that $E:=F+GL$ has all eigenvalues with absolute value less than 1, and let V be a neighborhood of zero such that Ex is in U for each x in V and such that using $u:=Ex$ on S' yields a system which is a.s. when starting at x in V. (Such a V exists by the argument in (3.3), since (F,G) is the linearization of p; an alternative proof would work directly on S using the Bellman–Gronwall lemma as in Desoer [2, p. 153].) Since z is a.c. in S, for each x in Z there is some input function $u(\cdot)$ and a T such that $x(t)$ is in V for all $t\geqslant T$. Since V is open, one may approximate such a $u(\cdot)$ by piecewise constant inputs in any fine enough partition of $[0,T]$, so for each x there is some input constant on intervals of the form $[n2^{-i},(n+1)2^{-i}]$ controlling x, and hence also a neighborhood of x, into V, in a time T that can be chosen as an integral multiple of 2^{-i}. By compactness of Z, there are some i and T which work for all x in Z. Without loss of generality one may assume that $i>j$. Take $d:=2^{-i}$.

Claim: The discrete-time system S'' obtained by sampling S at rate d has Z a.c. Indeed, if x is in Z the above construction shows that x can be taken into V using d-sampled inputs, i.e., as a state of the system S''. But once in V one may use the inputs $u(t+i):=Ex(t)$ (for $i=0,\cdots,m-1$, and t a multiple of $m=2^{i-j}$) to drive x asymptotically to zero. (Note that the mth iterate of the transition map p'' of S'', with "piecewise constant" inputs u as above, is by construction the same as p.) Thus, Z is a.c. The linearization (F',G') of p'' at zero is also a.c., because the lower frequency sampling (F,G) already is.

Thus, (3.3) applies to give a K which stabilizes Z for S'', which is equivalent to stabilizing Z in S with the d-sampled feedback K. ##

Remark 3.9: a) The assumption of continuity on f can be weakened considerably. All that is really needed is that sampled systems obtained from S satisfy (3.3), and this will sometimes happen under piecewise continuity of f. To make this precise would require entering the topic of defining "solution" of o.d.e.'s with discontinuous right-hand side.

b) For computations, the above proof is, of course, rather inefficient, and directly approximating S by a PL system may be more practical; this is yet another area requiring more careful study. Although of a very different nature, this result has some analogies with the piecewise-analytic theory of Sussmann [14]; these analogies are explored in Sontag and Sussmann [13], where it is also shown that, in general, a "piecewise" smoothness of control laws is unavoidable.

C. (Asymptotic) Regulators

The notions of weak and strong regulation introduced in Section II-C can be generalized in the obvious ways to the asymptotic case; Z should now be required to be a.s. instead of finite-time stable. This applies also to non-PL systems, as defined in A and B above. (Output maps are defined in the continuous case just as with discrete time.) In the continuous case, one looks for sampled regulators. As in Section II-C, S will be a state-output system.

Definition 3.10: Let S be a discrete-time system. The subset Z of X is *weakly regulable* iff there exists a system S' and a state x' in X' such that Z is a.s. [i.e., the coordinate $x(t)$ satisfies the conditions in (3.1)] for the closed-loop system when S' starts at x'. If Z is a.s. for S' starting at *any* state in X', then Z is *strongly* regulable. For a continuous-time system S, the subset Z is (weakly or strongly) regulable iff there is a discrete-time system obtained from S by sampling for which Z is (weakly or strongly) regulable. The system S is (weakly or strongly) *regulable on compacts* iff the above properties hold for each compact subset Z of X.

Variations of the definition are possible which would require S' itself to be stable. The constructions given below will provide this stability in any case, but the relations between the various possible definitions should be studied in detail in the future. Of course, the interest will be in "nice" (here, PL) regulators, but the above definition permits posing the question of whether such nice regulators exist when only the abstract possibility of regulation is given. Note that a regulable linear system (F,G,H) is one for which (i.e., (F,G) is stabilizable and (F,H) is detectable. Since linear systems appear, it will be convenient to assume that $q(0)=0$; this can always be done up to coordinates in Y.

Recall that two states x, x' of a system S are *indistinguishable* iff $h(x,w)=h(x',w)$ for every admissible (finite length) input function w. (The notations are as in Section II-B: in continuous time, $P(x,w)$ is the state reached after application of w.) "Admissible input function" means *any* sequence in discrete time, and any piecewise continuous w in continuous time. (Values $(w(t)$ must, of course, belong to U.) If Z is a subset of X, a coset for the indistinguishability (for simplicity, "indy") equivalence relation restricted to Z is an *indistinguishability class rel* (=relative to) Z. A set which is a union of such classes, for a given Z, is *saturated rel* Z. As a final definition, an input function w of finite length (final-state) *determines* the states of Z *up to indistinguishability* (u.t.i.) iff for any x in Z the outputs $H(x,w)$ are enough to determine the indy class of $P(x,w)$; a *sampled* such function w will be a w which is constant on intervals $[id, (i+1)d]$ for some d, *and* such that just outputs at times id (not greater than the length of w) are sufficient for this determination.

Assume now that S is weakly regulable on compacts. Let L be an indy class rel Z, some Z compact. Let S' be a weak regulator for Z, and let S start at $t=0$ in a state x belonging to L. From (2.12) it is clear that the future inputs to S will be the same if S starts at a different x' in L. Call $w(\cdot)$ the (infinite) input function thus obtained, which depends only on L and not on the particular x. The solutions $x(t)$ starting at the x in L satisfy the conditions in (3.1) for the given Z. Since L is compact (because indy classes are always closed by continuity of p,q), the convergence to zero is *uniform* on L, i.e., for any neighborhood V of zero there is a T such that $x(t)$ is in V for all $t\geq T$ and all $x(0)$ in L. This can be achieved by first covering L by neighborhoods W as in (3.1) and then taking a finite subcover. (In fact, although with different w's, a T can be obtained for the same reason as depending only on the compact Z.) Note that the existence of such $w(\cdot)$ depending only on L would be implied even if S and S' were both continuous-time systems and (2.12) would be suitably modified; this is all the more interesting in view of the fact that this condition turns out to be equivalent to the existence of a (sampled) regulator, and in fact a PL one.

Theorem 3.11: Let S be either a discrete- or a continuous-time system, with X, U, open, PL, and connected, and with p, q continuous, and differentiable in a neighborhood of zero. Assume that a) for every compact Z and every indy class L rel Z there exists an input function $w(\cdot)$ such that solutions $x(\cdot)$ starting at states in L converge uniformly to zero; b) the linearization (A,B,C) of S at zero is regulable; and c) for every compact Z there exists a sampled v which determines the states of Z up to indy.

Then S is weakly regulable (by PL systems). Further, if (p,q) are real-analytic and the local condition b) holds, then a) alone is, in fact, a necessary and sufficient condition for weak regulability.

Proof: The first part of the proof basically repeats the arguments in (3.3) and (3.7). More details are provided for the continuous-time case; discrete time is analogous but simpler. Let k be such that the linear system obtained by sampling (A,B,C) at rate 2^k is discrete-time regulable. From the linear theory one concludes the existence of a linear regulator for the latter, say $S''=(A'',B'',C'',D'')$, where S'' is built by cascading a Luenberger observer and a constant feedback matrix that provides asymptotic stability. Denote the state space of this regulator by $\mathbf{R}^n$, and states by x''; this S'' feeds $C''x''+D''y$ to (the sampled version of) (A,B,C). By the argument in (3.4), applied to the closed-loop system which consists of the sampling of S at that rate and of S'', there exists an open bounded PL neighborhood of zero D^* in $X\times\mathbf{R}^n$ such that a) D^* is invariant under closed-loop dynamics when the (sampling of) the original system is used together with S''; and b) there is asymptotic stability for the latter nonlinear closed-loop system when starting in D^*.

Moreover, this stability is preserved under higher rate sampling. For later use, note that there exists also a smaller set E^* with the same properties and such that the closure $\mathrm{cl}(E^*)$ of E^* is included in D^*. (This can be proved simply by starting in (3.3) with the box of sides $a/2$ for "V" instead of a.) Let the open subset D of X be such that $D\times\{0\}$ is included in D^*, and let F be a compact subset of $\mathbb{R}^n$ such that the projection of D^* in the second coordinate is included in F. Again for future reference, find E such that $E\times\{0\}$ is included in E^* and such that the closure of E is included in D.

Let Z now be a compact subset of X. We want to obtain a regulator S' for the states of Z. By c) there exists a sampled input v which determines states of Z u.t.i.; rescaling time if needed, we may assume that the sampling rate for v is 2^i for some i. (In discrete time, just let $i=0$.) Let i be the largest of i and the above k. By continuity of $P(\cdot,v)$ the set $Z':=P(Z\times\{v\})$ is again compact.

Consider now an indy class L rel Z', and a corresponding w as in a). Let T be an integer such that $x(t)$ is in D for all $x(0)$ in L and all $t\geqslant T$. Let x be in L. Consider the map (a restriction of P) P^T: $X\times U(T)\to X$, which assigns final states $x(T)$ to initial states $x(0)$ and piecewise continuous input sequences of length T. With the uniform convergence topology on $U(T)$, P^T is continuous. Since $P(x,w)$ is in D, there is then a neighborhood $A\times B$ of (x,w) which is also mapped into D. For each x in L find such an $A(x)$ and $B(x)$. Since the $A(x)$ then cover the compact L, there is a finite subcover, say $A(x_1),\cdots,A(x_n)$. Let A be the union of the $A(x_i)$, and let B be the intersection of the $B(x_i)$. Pick any w' in B. For some j, there is such a w' which is moreover sampled at rate 2^j. Thus, $P(x,w')$ is in D for any x in A. Replacing w by w', one concludes that there is a *sampled* w controlling L (and a neighborhood A of L) into D at time T. Let A' be the intersection of A and Z'.

The set A' (open in Z') can be assumed to be saturated rel Z'. Indeed, consider the quotient space Z'/I, where I is the indy relation, and let θ be the canonical quotient map. Since Z' is compact and Z'/I is Hausdorff (by continuity again), θ is closed. So the complement of $\theta^{-1}\theta(Z'/A)$ is open in Z', saturated rel Z', contained in the original A', and still contains L, as required.

For each class L there are then a sequence w, an integer T, and a neighborhood A with the above properties. By compactness of Z', one concludes that there exist finitely many, say m, open sets A_j and corresponding A'_j saturated rel Z', and input functions w_j of lengths $T(j)$, all sampled at some rate 2^{-k}, such that $P(x,w_j)$ is in D whenever x is in A_j. Replacing i by the largest of i and k, a fixed sampling rate 2^i will be used for the input v which determines states of Z u.t.i. (in the process driving Z to Z'), the inputs w_j which control Z', and the sampling rate for (A,B,C) for which S'' regulates locally.

From now on, the problem is strictly one about discrete-time systems: for the continuous-time case a regulator will be constructed for the system obtained sampling at the above rate. Let r be the length of v and let k be the largest length of the sequences w_i. When sampling as above, one would get $k=$ largest of the lengths $k(j)=2^iT(j)$. Consider the set

$$\underline{C}:=\{(H(x,v),[P(x,v)])|x \text{ in } Z\}\subseteq Y^r\times(Z'/I) \tag{3.12}$$

where $[-]$ indicates indy class (restricted to Z'). Since Z' is compact and P, H, are continuous, $\underline{C}$ is also compact. Since v determines states of Z u.t.i., $H(x,v)$ uniquely determines $[P(x,v)]$, so $\underline{C}$ is the graph of a continuous function f. Denote by J the domain of f. Let W_j be any open set in Y^r interesting J at $f^{-1}([A'_j])$. (Note that saturation of A'_j gives precisely that it is open in the quotient space.) If x is any state in Z whose output $y(\cdot)$ (when v is applied) is in J, then $y(\cdot)$ is in some W_j, and for that j the corresponding w_j controls $P(x,v)$ into D. Replace now each W_j by a subset W'_j which is PL, in such a way that the latter still cover J. (This can be done using that the W_j are open and compactness of J.) Replacing by new sets W''_j one obtains a *disjoint* PL (nonopen) subcover of J. Let g be the PL map which assigns the integer j to each W''_j.

The construction of the regulator S' is now easy. The state set of S' will consist of the product

$$Y^{r-1}\times N(r+k+1)\times N(m)\times\mathbb{R}^n \tag{3.13}$$

where $N(j)$ is the set of integers $\{1,\cdots,j\}$. (The compact set F could have been used instead of the last coordinate.) To describe the dynamics of S', denote by i, j, y'', x'' the contents of $N(r+k+1)$, $N(m)$, Y^{r-1}, and $\mathbb{R}^n$ respectively. The operation shift (a,y'') stores the value a in the last coordinate of y'' and shifts the rest of y'' to the left, dropping the first value. The input is the present output value y of S (sampled at the beginning of the period in the continuous-time case), and the output is the value that results in the register u. Finally, the vector $k(j)$, the function g, the linear system S'', the vector $v(i)$, and the matrix $w(i,j)$ [containing $w_j(i)$] are assumed to have been already defined. Then S' evolves according to the following:

$$\begin{aligned}
&u:=\text{if } i\leqslant r \text{ then } v(i)\\
&\qquad \text{else if } r<i\leqslant k(j) \text{ then } w(i-r,j)\\
&\qquad \text{else } C''x''+D''y;\\
&\text{if } i=r \text{ then } j:=g(y'',y);\\
&\text{if } i\leqslant r+k \text{ then } i:=i+1;\\
&y:=\text{shift}(y'',y);\\
&\text{if } i>r+k(j) \text{ then } x'':=A''x''+B''y.
\end{aligned} \tag{3.14}$$

The initial state is $i:=1$, j arbitrary, y'' arbitrary, and $x'':=0$. It is clear that (3.14) defines a PL system. By construction, any state of Z is first sent into a known state in Z', and then driven to the domain of local a.s.; thus S' is indeed a weak regulator.

It does remain to prove that condition c) is not needed in the analytic case. Since the proof is basically that in Sontag [11, Section 3], it will only be sketched here. For any input function w let $K(w)$ be the set of pairs of states x, x' of Z with $h(x,w)=h(x',w)$. Each $K(w)$ is compact and defined by analytic equations. For each finite set $\{w_1,\cdots,w_r\}$, let $K(w_1,\cdots,w_r):=$ intersection of $K(w_1),\cdots,K(w_1,\cdots,w_r)$. Consider the subclass consisting of all those sets of type $K(w_1,\cdots,w_r)$ which use only input sequences w_j which are sampled at rate 2^i, for all possible i and r. Let $K(w_1,\cdots,w_r)$ be such a minimal set. Assume that the outputs to $v=w_1\cdots w_r$ at the end of w_1, w_1w_2, etc., would not determine states of Z u.t.i. There are then two states x, x' in Z which give the same output at those times under v, i.e., such that (x,x') is in $K(w_1,\cdots,w_r)$, but such that $(a,a'):=(P(x,v),\ P(x',v))$ is not in I. Let w be an input with $h(a,w)\neq h(a',w)$. By continuity of output values on time, w may be assumed to be of some length $k2^{-i}$ for some k, i. By continuity of output values on input functions, w may be assumed to be sampled at some rate 2^i. Thus, the set $K(w_1,\cdots,w_r,w)$ is strictly contained in $K(w_1,\cdots,w_r)$, contradicting minimality of the latter. Thus, v indeed determines states of Z u.t.i., as a sampled input. ##

For strong regulation the situation is more complicated, but the result given below appears to be nearly necessary and sufficient at least in the continuous-time analytic (or in the discrete-time but polynomial) case: in that case it is generally known that a generic set of inputs (relative to a suitable topology) exists, each of which serves to determine final states u.t.i.; it is not known (and in fact it is false) that this set should contain a neighborhood of zero, or that sampled inputs may be used, and these facts will be needed below. (It is possible, however, that the stronger facts may still hold in the analytic discrete and continuous cases, at least when the state space is bounded and requiring only "almost" indy u.t.i., which is enough for the proofs.) For simplicity, the result will be proved under an observability assumption. When dealing with "smooth" systems observability may be expected to be a generic condition on systems (transition maps transversal to "kernel" of output maps, intuitively)—a rigorous version and proof of this has in fact been shown to the author by H. Sussmann. A compactness assumption on U is also made for simplicity; mathematically, this is a nontrivial assumption, but from a practical viewpoint it is, of course, very natural [this is dropped in (3.23)]. Call a subset V of the state space X *globally observable* iff every indy class in X has at most one element in V. The topology on input functions is taken to be that of uniform convergence, and "generic" will mean a countable intersection of open dense sets. Assume that U is compact and contains a neighborhood of zero.

Theorem 3.15: Let S be as in (3.11), satisfying a) and b) there, as well as: c)′ there exist an integer M and a generic subset $\underline{U}$ of the set of input functions of length M such that $\underline{U}$ contains all functions with values in a neighborhood G of zero and such that each sampled w in $\underline{U}$ determines the states of X u.t.i.; and c)′ there is a globally observable neighborhood V of the zero state. Then S is strongly regulable (by PL systems).

Proof: Let k, S'', D, etc., be as in the first paragraph of the proof of (3.11). Assume without loss of generality that V is an open PL set, and that all values $C''x''+D''q(x)$ are in G when (x,x') is in D^*. Construct all of these in such a way that the closure of the projection F' of D^* in the first factor is included in V. Let Z be a compact, and assume without loss that it contains F'. Let Z'' be a compact set containing the union of all the images $P(Z\times U(t))$ (where $U(t):=$set of input functions of length t), for $0\leqslant t\leqslant 2M$, and let Z' be another compact containing all the images $P(Z''\times U(t))$, for $0\leqslant t\leqslant M$. (Thus, Z' also contains all states reached from the original Z in at most time $3M$.)

Repeat now the constructions of the covering $\{A_j\}$ and the sequences w_j, *but* regarding controllability to E as opposed to controlling only to D. Moreover, pick the final w_j corresponding in the proof to each class L as a sequence such that either $T(j)<M$ or every length-M sampled subsequence of w_j determines states of u.t.i. (The latter requirement will be met if one can prove that length-M sequences with that property form a dense set, since B will then contain at least one such sequence. But the set of wanted sequences is given by a finite intersection of generic sets—the sampling having been already chosen—and hence still dense.) Use the notations $k(j)$, k, m as earlier, and $r:=2^iM$. Construct the set $\underline{C}$ using the constant input $v=0$ of length r (any other small enough v would do), *but* using Z'' whenever Z was used for (3.12). The function g is constructed as before.

The above constructions can be combined as before to give a weak regulator which first determines the state u.t.i. and then drives this state to the domain of regulation for S''. Recall from Part II that the idea of strong regulation is intuitively that sudden disturbances during the operation of this closed-loop system should not affect eventual stability. One way to ensure this stability is to constantly check (using a PL system!) for deviations of the observed I/O behavior from the "expected" behavior obtained when no disturbances occur. This checking will be accomplished as follows.

Let v_{ij} be the length-r input sequence obtained as the restriction of w_j to (sampled instants) $t=i,\cdots,i+r-1$. Thus, v_{ij} is defined for $j=1,\cdots,m$ and $i=1,\cdots,k(j)-r+1$ when $k(j)$ is at least r. Note that each v_{ij} determines states of Z' u.t.i. For each $0\leqslant i\leqslant k(j)$, let w'_{ij} denote the tail subsequence obtained by restricting w_j to its last $k(j)-i+1$ values. Let D_{ij} [respectively, E_{ij}] be the set of states of Z' that w_{ij} sends into D [respectively, E]. Each D_{ij} and E_{ij} is saturated rel Z'. Indeed, let x be in D_{ij} and let x' be indistinguishable from x. Then $P(x,w_{ij})$ and $P(x',w_{ij})$ are indistinguishable. But $P(x,w_{ij})$ is in D, which is included in the globally observable V; thus $P(x,w_{ij})=P(x',w_{ij})$ and x' is also in D_{ij} (clearly, much less than global observability of V would be needed here, since all that one really wants is that $P(x',w_{ij})$ be again in V). The same argument applies to E_{ij}. Their closures are also saturated, and by the construction of D, E, $\text{cl}(E_{ij})$ is contained in

D_{ij}. Consider when $k(j) \geq r$ the sets

$$\underline{C}_{ij} := \{ (H(x, v_{ij}), [P(x, v_{ij})]) | x \text{ in } \mathrm{cl}(D_{ij}) \}. \quad (3.16)$$

Since the v_{ij} determine u.t.i., each such $\underline{C}_{ij}$ defines an f_{ij} as in (3.12). Let $\boldsymbol{E}_{ij}$ [respectively, $\boldsymbol{D}_{ij}$] be the (open set in the domain of f_{ij}) preimage of $E_{i(j+r)}$ [respectively, $D_{i(j+r)}$]. Note that $\mathrm{cl}(\boldsymbol{E}_{ij})$ is included in $\boldsymbol{D}_{ij}$.

There is, then, a closed PL subset $L(i, j)$ of Y^r which contains $\boldsymbol{E}_{ij}$ and is contained in $\boldsymbol{D}_{ij}$. Further, the $L(i, j)$ can be constructed, for each j, satisfying certain compatibility conditions for different i. Namely, that

$$p(f_{ij}(L(i, j) \times \{w_{ij}\})) \subseteq f_{i+1,j}(L(i-1, j)). \quad (3.17)$$

To obtain this, start by defining $L(i, j)$ for $i=1$; next consider the compact set $P(f_{ij}(L(i, j)) \times \{w_{ij}\})$; this is included in D^* and includes E^*, so in constructing the next set one may require containing this set, then repeat for larger i. Let g_{ij} be the (PL) characteristic function of $L(i, j)$. Then, if $g_{ij}(H(x, v_{ij}))$ is 1 it follows that $P(x, w_{ij})$ is in D, and conversely if $P(x, w_{ij})$ is in E one knows that this value is one.

The above functions will be used to detect deviations while the w_j are being applied. But disturbances may also happen while in the domain of local linear regulation. For this case, consider the set

$$\{ (u, H(x, u), [P(x, u)]) | x \text{ in } F', u \text{ in } U^r \} \quad (3.18)$$

(recall that F' includes $\mathrm{cl}(pr_1(D^*))$). This defines on its domain a continuous function θ such that $\theta(u, y)$ gives $[P(x, u)]$ in X/I for the uniquely determined class. Consider the product space $(X/I) \times \mathbb{R}^n$ (the second factor is the linear regulator state). Since I is trivial on V, the set $V \times \mathbb{R}^n$ can be identified with a subset of this product space, and hence also E^* and D^* are subsets of it. Let $\theta'(u, y, x'')$ be $(\theta(u, y), x'')$. Since $\mathrm{cl}(E^*)$ is included in D^*, there exists a PL subset of $U^r \times Y^r \times \mathbb{R}^n$ which maps under θ into a set containing E^* and contained in D^*; let c be its characteristic function. Thus, at any time $c(u, y, x'')$ is in particular 1 when (x, x'') is in E^* for the unique x giving past I/O data (u, y), and conversely if this value is 1 then at least (x, x'') is in D^*.

The construction of S' is now as follows. The state set is

$$X' := U^{r-1} \times Y^{r-1} \times N(r) \times N(m) \times N(k) \times \mathbb{R}^n \times X_0 \quad (3.19)$$

with notations as earlier and X_o a three-element set {TEST, CONTROL, STANDBY}. To display the dynamics, denote by z the coordinate of X_o and denote y'', x'' as before, u'' the content of the U^{r-1} factor, $w(i, j)$ as before, i, j, k' the contents of the $N(r)$, $N(m)$, and $N(k)$ registers respectively, $g'(i, j, y)$ for $g_{ij}(y)$, and let INPUT(u) be false iff u is in V. All partially defined maps are assumed extended arbitrarily outside their domains. The dynamics are

(z=TEST:)u:=0;

  if $i<r$ then i:=i+1 else

   begin j:=$g(y'', y)$; k':=1; z:=CONTROL; end;

   y'':=shift (y'', y); u'':=shift (u'', u);   (3.20)

(z=CONTROL:)u:=$w(k', j)$;

  if $k'<k(j)$ then k':=k'+1 else

   begin z:=STANDBY; x'':=0; end;

  if $r \leq k'$ then

  if $g'(i, j, y'', y)=0$ then z:=TEST;

  y'':=shift (y'', y); u'':=shift (u'', u); (3.21)

(z=STANDBY:)u:=if INPUT (u) then 0

   else $C''x''+D''y$;

  x'':=$A''x''+B''y$;

  if (INPUT (u) OR $c(u'', y'', x'')=0$)

   then z:=TEST;

  y'':=shift (y'', y); u'':=shift (u'', u); (3.22)

To prove that S' is a strong regulator, let x' be an arbitrary initial state of S' and let x be in Z. Consider the closed-loop evolution. Say that *the alarm sounds* if at some time during the evolution z changes from CONTROL or STANDBY into TEST.

Assume that the alarm does sound. Then it must sound at a time when x is in Z''. This is proved as follows. If originally z=TEST, then after at most r instants one has that z=CONTROL. Assume that the alarm sounds while z=CONTROL. Let k' be the (first) time it sounds. If $r \leq k'-1$, then at $t-1$, S' had decided that $x(t-1)$ will be driven into D by w_j (which is being applied) and will hence converge asymptotically to zero. The condition in (3.17) ensures that the alarm will not sound again. So $k'-1>r$, i.e., $k' \leq r$, and so at most $2r$ steps have elapsed since the initial time. Thus, $x(k')$ is indeed in Z''. If the alarm does not sound while z=CONTROL it means that $k(j)<r$ (otherwise the argument above ensures it would never sound). Then at most $2r-1$ steps have elapsed by the time the state enters z=standby. If the alarm does sound at the first time that z=STANDBY then $t \leq 2r$ and again $x(t)$ is in Z''. Otherwise, it will never sound, since c ensures that (x, x'') is in the invariant a.s. set D^*. If the alarm does not sound, the above arguments show that one indeed has a.s., since $x(t)$ is sent to D^* and the convergence is uniform once there. If the alarm did sound, z is set to TEST, and after r more steps the state of S is determined u.t.i., and a suitable j is obtained with convergence to E thereafter (the alarm does not sound again, since x is sent into E, not just D, and $E \times \{0\}$ is in E^*). ##

Various modifications of the above would be desirable. For example, the replacement of all the equality decisions by tolerance comparisons would be needed in order to insure robustness of the design. We leave this as a point to be further studied, and concentrate instead in weakening the hypotheses of (3.15). The proof of the following variant is (except for a more complicated notation) basically the same as for (3.15), and hence will be omitted. Note that compactness of U is not required anymore.

Theorem 3.23: Let S be as in (3.15) and assume that 1) the linearization of S at zero is regulable; 2) there is a globally observable neighborhood V of the zero state; and 3) for each compact Z there exist a compact neighborhood U_o of O in U, a positive M, and a generic subset $\underline{U}$ of $U_o(M)$ such that the following properties hold: i) for some compact Z' containing all $P(Z\times U_o(t))$, $0\leqslant t\leqslant 3M$, and for each indy class L rel Z', there is an input function driving states in L uniformly to zero; and ii) there is a sampling time d (dividing M) such that any input in $\underline{U}$ which is sampled at a rate divisible by d^{-1} serves to determine states of Z' u.t.i. Then S is strongly regulable. ##

In the continuous-time case, the assumption on V implies observability. This can be weakened by asking directly if there exists a linear regulator which converges when starting at states indistinguishable from those close to the origin.

Example 3.24: The hypotheses of (3.23) seem to be quite weak. As a very simple illustration, consider the one-dimensional, single input system $(dx/dt)(t)=x(t)+u(t)$, $y(t)=x(t)+x(t)^2$. Clearly, 1) above is satisfied, while 2) follows from observability. Consider the sampled system (rate d^{-1}): $x(t+1)=ax(t)+bu(t)$, $y=x+x^2$, where $a=e^d$, $b=a-1$. Assume that input u is applied to this system in state x; then $y_1=x+x^2$ and $y_2=ax+bu+(ax+bu)^2=(\cdots)+(a-a^2+2abu)x$ are observed, where $(\cdots)$ depends only on y_1 and u. Thus, x will be obtainable from y_1, y_2 when $a-a^2+2abu\neq 0$, i.e., when $u\neq 1/2$. Thus, inputs of length 2 are always sufficient for determining states of the sampled system, as long as the value $1/2$ is not used. We shall now choose both d and $M:=2d$. Let Z be a compact, say, contained in $[-k,k]$. A suitable U_o (to be taken of the form $[-sk,sk]$) and a d should be such that inputs of magnitude sk suffice for controlling states reached in time $3M=6d$ from Z. In other words, one needs that $|e^{6d}x+(e^{6d}-1)u|<sk$, where $|x|\leqslant k$, $|u|\leqslant sk$. Thus d, s must satisfy $ke^{6d}+sk(e^{6d}-1)<sk$, or $6d<\ln(2s/(s+1)$. For example, $d=0.04$, $s=2$ will satisfy the above. (In fact, since s can be taken as close to 1 as wanted, this means that the "transient" state set Z' can be made to be as close to Z as wanted, as long as the sampling rate is high enough. Obviously, in a practical situation too fast a sampling will lead to numerical instability.) ##

IV. Conclusions and Remarks

After an introduction and basic terminology, Section II studied bounded time problems. The study there consisted basically in showing that certain "external" (and nonfirst-order) properties of systems are equivalent to natural "internal" (first-order) properties, resulting then in PL solutions for regulators. Section III concentrated on properties which are not purely algebraic, although the PL structure was also very useful here, and in the design of PL regulators for non-PL systems.

The previous material should be taken mostly as an illustration of the possibility of developing an approach to nonlinear regulation using PL systems. Although various results were given, a much greater number of questions, some of which were already mentioned, remain open. This applies both to problems of computation and to problems of a more theoretical nature. Among the latter we wish to suggest a few other areas not treated above. One of these is the topic of optimization. It is easy to see that finite-time optimal control problems with PL costs (e.g., absolute-value norms) result in closed-loop PL feedback solutions. In fact, even a "piecewise quadratic" theory could be developed, resulting in PL solutions. For infinite time problems, an approach much like that in the last section (around the origin a linear-quadratic problem) should give a constant PL feedback. This should be worked out in more detail. The structural stability of the various constructions should be also explored carefully. Finally, questions involving uncertainty have not been treated at all, but it conceivable (although not clear!) that suitable stochastic control and filtering questions may be posed and solved in the present setup.

References

[1] L. O. Chua, *Introduction to Nonlinear Network Theory.* New York: McGraw-Hill, 1969.

[2] C. A. Desoer, *Notes for a Second Course on Linear Systems.* Princeton, NJ: Van Nostrand, 1970.

[3] T. Fujisawa and E. S. Kuh, "Piecewise linear theory of nonlinear networks," *SIAM J. Appl. Math.*, vol. 22, pp. 307–328, 1972.

[4] M. Gatto and G. Guardabassi, "The regulator theory of finite automata," *Inform. Contr.*, vol. 31, pp. 1–16, 1976.

[5] A. Gill, *Introduction to the Theory of Finite-State Machines.* New York: McGraw-Hill, 1962.

[6] M. L. J. Hautus, "Feedback in general discrete-time systems," Eindhoven Univ. Technol., Memo. COSOR 78-17, 1978.

[7] R. E. Kalman, Y. C. Ho, and K. S. Narendra, "Controllability of linear dynamical systems," *Contr. Differential Equations*, vol. 1, pp. 189–213, 1963.

[8] S. K. Kang and L. O. Chua, "A global representation of multidimensional piecewise linear functions with linear partitions," *IEEE Trans. Circuits Syst.*, vol. CAS-25, pp. 938–940, 1978.

[9] E. B. Lee and L. Markus, *Introduction to Optimal Control Theory.* New York: Wiley, 1967.

[10] N. Minorski, *Theory of Nonlinear Control Systems.* New York: McGraw-Hill, 1969.

[11] E. D. Sontag, "On the observability of polynomial systems, I: Finite time problems," *SIAM J. Contr. Opt.*, vol. 17, pp. 139–151, 1979.

[12] ——, "Remarks on piecewise linear algebra," *Pacific J. Math.*, to be published.

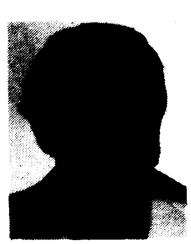
[13] E. D. Sontag and H. J. Sussmann, "Remarks on continuous feedback," in *Proc. IEEE Conf. Decision Contr.*, Albuquerque, NM, 1980.

[14] H. J. Sussmann, "Single input observability of continuous time systems," *Math. Syst. Theory*, vol. 12, pp. 31–52, 1979.

[15] ——, "Subanalytic sets and feedback control," *J. Differential Equations*, vol. 31, pp. 31–52, 1979.

[16] P. Vidal, *Nonlinear Sampled-Data Systems.* Gordon and Breach, 1969.

[17] M. Wonham, *Linear Multivariable Control.* New York: Springer, 1974.



Eduardo D. Sontag was born in Buenos Aires, Argentina, on April 16, 1951. He received the Licenciado degree in mathematics from the University of Buenos Aires, and the Ph.D. degree in mathematics from the Center for Mathematical System Theory at the University of Florida, Gainesville.

From 1976 to 1977, he held a postdoctoral fellowship at the University of Florida, Gainesville. In 1977, he was appointed Assistant Professor of Mathematics at Rutgers—The State Uni-

versity, New Brunswick, NJ. He also worked in nonacademic computer-related positions. His major research interests lie in mathematical system theory, applied algebra, and theoretical computer science. He is the author of various papers in system theory and algebra, and has also written survey papers and a book on artificial intelligence. He has authored *Lecture Notes in Control and Information Sciences, 13: Polynomial Response Maps* (Berlin: Springer, 1979).

Dr. Sontag is a member of Phi Beta Kappa, of the American Mathematical Society, and of the Society for Industrial and Applied Mathematics.