

ON DISCRETE-TIME POLYNOMIAL SYSTEMS

E. D. SONTAG*

Center for Mathematical System Theory, Department of Mathematics, University of Florida,
Gainesville, FL 32611, U.S.A.

and

Y. ROUCHALEAU*

Center for Mathematical System Theory, Department of Mathematics, University of Florida,
Gainesville, FL 32611, U.S.A.

and

Centre d'Automatique, E. N. S. Mines F-77305 Fontainebleau, France

(Received 1 March 1976)

Key words: Nonlinear dynamical systems, system analysis, reachability, observability, canonical systems, applied algebra.

INTRODUCTION

THE FIELD of Algebraic System Theory is slowly expanding beyond the traditional confines of linear systems to include various restricted types of nonlinear systems, for instance, bilinear systems. We shall consider here a more general type of systems which have algebraic constraints on their state set and for which the state transitions are given by (arbitrary) polynomial functions of the inputs and state variables. The only technical restriction we shall make is to consider systems which begin their evolution from an equilibrium state.

This paper is a summary (of a preliminary nature) of the results obtained so far on the above questions. A thorough study of the realization theory of polynomial systems is carried out in Sontag [7].

One of our main concerns in this work has been to obtain finiteness conditions, for it is our belief that finite computational procedures are an essential goal of applied algebra.

In the first four parts of this paper we give the necessary mathematical preliminaries and define the systems that we study. Part 5 is concerned with reachability in bounded time, while in Part 6 we look at the problem of deciding whether two systems have the same external behavior by applying finitely many inputs. In Part 7 we show that finitely many inputs (which can be chosen quite arbitrarily) are sufficient to separate those states of a system which are distinguishable. Part 8 is an investigation concerning the conditions under which systems with the same external behavior are isomorphic.

* This research was supported in part by US Army Research Grant DAHC04-74-G-0153 and by US Air Force Grant AFOSR 72-2268 through the Center for Mathematical System Theory, University of Florida, Gainesville, FL 32611, U.S.A. This is an expanded version of the paper with the same title presented at the CNR-CISM Symposium on Algebraic System Theory, June 1975, Udine, ITALY. The editor of the proceedings, G. Marchesini, was unable to include this paper in the proceedings.

1. ALGEBRAIC PRELIMINARIES

We review now some notions from commutative algebra and algebraic geometry. The results can be found in any of the standard references; see, for instance, Dieudonné [1], Mumford [4], Macdonald [3]. Although the literature usually treats only the case in which solutions are taken in algebraically closed fields, the generalization to the present setup (namely, arbitrary infinite fields) does not present any major difficulty. We shall therefore omit the proofs here.

Throughout this paper, k denotes an arbitrary but fixed infinite field. For any nonnegative integer n , k^n is the set of all n -tuples of elements in k . In particular, k^0 is a one-element set $\{0\}$. Similarly, $k[T_1, \dots, T_n]$, or $k[T]$ when there is no danger of confusion, is the ring of polynomials in the indeterminates T_1, \dots, T_n with coefficients in k . If $n = 0$, $k[T]$ is defined to be k . When f is in $k[T_1, \dots, T_n]$ and $x \in k^n$ then $f(x)$ denotes the evaluation (i.e. *specialization*) of f at x .

For each subset $Y \subseteq k[T_1, \dots, T_n]$ we define

$$V(Y) := \{x \in k^n \mid f(x) = 0 \text{ for all } f \in Y\}.$$

For each subset $S \subseteq k^n$ we define

$$I(S) := \{f \in k[T_1, \dots, T_n] \mid f(x) = 0 \text{ for all } x \in S\}.$$

Of course, $V(Y) = V(\langle Y \rangle)$, where $\langle Y \rangle$ denotes the ideal generated by the set Y . Since every ideal of $k[T]$ is finitely generated (Hilbert's Theorem), we note that

Lemma 1.1. For every $Y \subseteq k[T]$ there exists a finite subset $Y_0 \subseteq Y$ such that $V(Y) = V(Y_0)$.

Proposition 1.2. Let V and I be defined as above. Then V, I give a Galois connection between the lattices of subsets of $k[T_1, \dots, T_n]$ and of k^n . Furthermore, the induced closure operator $S \mapsto \bar{S} := V(I(S))$ defines a topology on k^n , called the *Zariski topology*. The closed sets for this topology are called *k -points of algebraic sets*. Algebraic sets in k^n can themselves be considered as topological spaces under the topology induced from k^n .

Since the field k is infinite, a *polynomial function* $f: k^n \rightarrow k: x \mapsto f(x)$ can be identified in an unambiguous way with a polynomial in $k[T_1, \dots, T_n]$. We shall always assume such an identification and shall speak indistinctly of polynomials and polynomial functions. A polynomial function $f: V \rightarrow k$, where $V \subseteq k^n$, is by definition the restriction to V of a polynomial function $f: k^n \rightarrow k$.

Definition 1.3. Let $V \subseteq k^n$ be an algebraic set. Then the k -algebra $A(V)$ of all polynomial functions $f: V \rightarrow k$ with pointwise operations is the *coordinate ring* of V . The k -algebras of type $A(V)$ are *affine k -algebras*.

It is not difficult to verify that $A(V) = k[T]/I(V)$. When k is algebraically closed, affine k -algebras are characterized as finitely generated k -algebras with no nilpotent elements.

The *dimension* of a k -algebra A is defined as the maximum possible length of chains of prime ideals. When A is an integral domain finitely generated over k , the dimension of A coincides with the transcendence degree of A over k .

We say an algebraic set $V \subseteq k^n$ is *irreducible* if there do not exist algebraic sets V_1, V_2 such that $V = V_1 \cup V_2$ and $V \neq V_1, V \neq V_2$. Such algebraic sets are also called (affine) *varieties*. The following results are well known.

Proposition 1.4. An algebraic set V is irreducible if and only if $I(V)$ is a prime ideal. If $\dots \subset V_{i-1} \subset V_i \subset V_{i+1} \subset \dots$ is a strictly monotonic chain of algebraic sets, then $\dots \supset I(V_{i-1}) \supset I(V_i) \supset I(V_{i+1}) \supset \dots$ is also strictly monotonic. In particular, every ascending or descending chain of irreducible algebraic subsets of V has a length at most equal to the dimension of $A(V)$. The dimension of $A(k^n)$ is n .

We remark that the last statement of Proposition (1.4) is the only place where the assumption $k = \text{field}$ is essential. Generalizations to arbitrary integral domains are not very difficult.

The (*algebraic*) *dimension* $\dim V$ of an algebraic set V is by definition the dimension of $A(V)$. Except for the algebraically closed case, this number is only an upper bound on the *geometric dimension* given by the maximum length of chains of varieties included in V . Clearly, if $V \subseteq k^n$ then $\dim V \leq n$.

Definition 1.5. Let V, W be algebraic sets and $W \subseteq k^m$. A map $f: V \rightarrow W$ is called a *morphism* (or *polynomial map*) if there exist f_1, \dots, f_m in $A(V)$ such that $f(x) = (f_1(x), \dots, f_m(x))$ for all $x \in V$. The associated *comorphism* $A(f): A(W) \rightarrow A(V)$ is the k -algebra homomorphism defined by $A(f): g \rightarrow g \circ f$ for all $g \in A(W)$.

Theorem 1.6. Algebraic sets and morphisms define a category, which is equivalent to the dual of the category of affine k -algebras and k -algebra homomorphisms.

Remark 1.7. The image of an algebraic set under a morphism is not necessarily an algebraic set. However, it can be proved (see Mumford [4], Chapter I, §8, Corollary 2) that *the image of an algebraic set is a constructible set if k is an algebraically closed field.* (A set is *constructible* in V if it can be expressed as a finite union of sets of the form $U \cap F$, where U is open and F is closed in the Zariski topology of V .)

Theorem 1.8. Let $f: V \rightarrow W$ be a morphism of algebraic sets. Then:

- (a) f is continuous for the Zariski topologies on V and W .
- (b) f is dominating, i.e. $\overline{f(V)} = W$, if and only if $A(f)$ is injective.
- (c) f is injective if $A(f)$ is surjective.

Remark 1.9. Injectivity of f does not imply surjectivity of $A(f)$, except in special but important cases, for example, when f is the restriction of an injective *affine* map $f: k^n \rightarrow k^m$. (Surjectivity of $A(f)$ is in fact equivalent in general of f being a *closed embedding*.)

Proposition 1.10. Products exist in the category of algebraic sets. Moreover, $A(V \times W)$ is isomorphic as a k -algebra to $A(V) \otimes A(W)$, the tensor product of the k -algebras $A(V)$ and $A(W)$.

Intuitively, if $V \subseteq k^n$ and $W \subseteq k^m$ then the categorical product of V and W is the set product $V \times W$ seen as an algebraic subset of k^{n+m} .

From Hilbert's theorem on ascending chains, it is known that a chain of ideals in $k[T]$ is necessarily finite. A recent theorem gives explicit bounds for the lengths of such chains:

Theorem 1.11. (Seidenberg [5]). Let $\kappa(j)$ be a nonnegative integer for $j = 0, 1, \dots$ and consider strictly ascending chains of ideals $I_0 \subset I_1 \subset \dots \subset I_r$ in $k[T_1, \dots, T_n]$, where I_j has a basis of elements of degree $\leq \kappa(j)$. Then there is an integer $\mu(\kappa, n)$ computable from κ and n such that the length of any such chain is smaller than $\mu(\kappa, n)$.

For the explicit form of μ , the reader is referred to the work of Seidenberg. Another way of expressing the conclusion of the theorem is the following. We call an (infinite) sequence of ideals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ strict if it satisfies the property:

$$\text{If } I_n = I_{n+1} \text{ then } I_{n+1} = I_{n+2}.$$

Then the conclusion of (1.11) for a strict sequence is that if each I_j can be generated by polynomials of degree $\leq \kappa(j)$ then $I_s = I_{s+1}$ for $s = \mu(\kappa, n)$.

Finally, we shall need an interpolation result. If $f \in k[T_1, \dots, T_n]$ then $\deg_i f$ is by definition the degree of f as a polynomial in T_i .

Proposition 1.12. Assume $L \subseteq k$ is a subset of cardinality $m + 1$. Let $\Delta: L^m \rightarrow k$. Then there

exists a unique $f \in k[T_1, \dots, T_n]$ such that $\deg_i f \leq m$ and $f(l_1, \dots, l_n) = \Delta(l_1, \dots, l_n)$ for all $(l_1, \dots, l_n) \in \mathcal{L}$.

2. SYSTEMS

Let k be an infinite field, Y a fixed algebraic set over k and U a fixed variety (irreducible algebraic set). Then

Definition 2.1. A (polynomial, discrete time, constant) dynamical system over k , having input space U and output space Y , is $\Sigma = (X, x_0, p, h)$, where

- (i) X (the state space) is an algebraic set over k .
- (ii) x_0 (the initial state) is a fixed element of X .
- (iii) p (the transition map) and h (the output map) are morphisms $X \times U \rightarrow X$ and $X \rightarrow Y$.

Remarks 2.2. (i) $x_t = p(x_{t-1}, u_{t-1})$ is interpreted as the state at time t corresponding to the state x_{t-1} and the input u_{t-1} at time $t - 1$; $y_t = h(x_t)$ is interpreted as the output at time t .

(ii) Since k^m is irreducible, a system with unconstrained inputs fits in our framework.

(iii) Since any algebraic set is embedded in some affine space, there is no loss of generality in assuming that the output set is k^0 .

(iv) x_0 can be identified with a map from k^0 into X .

Definition 2.3. The dimension of a system Σ is that of the algebraic set X_Σ . If $X_\Sigma \simeq k^n$, the system Σ is said to be unconstrained.

The map p can be extended recursively to a map $X \times U^* \rightarrow X$ (where U^* is the free monoid on U), which we shall also denote by the letter p . Let $g: U^* \rightarrow X$ be defined by $g(\omega) = p(x_0, \omega)$. Then:

Definition 2.4. The input-output map of a system Σ is the map $f: U^* \rightarrow Y$ given by $f = h \circ g$.

We shall denote by g_t and f_t the restrictions of g and f to U^t (the t -fold Cartesian power of U). It follows from the definition of p and h that g_t and f_t are morphisms from the variety U^t into X and Y .

Definition 2.5. Let $\Sigma = (X, x_0, p, h)$ and $\hat{\Sigma} = (\hat{X}, \hat{x}_0, \hat{p}, \hat{h})$ be two polynomial systems. A morphism of systems $\Sigma \rightarrow \hat{\Sigma}$ is a morphism T of algebraic sets such that the following diagram commutes:

$$\begin{array}{ccc}
 X \times U \xrightarrow{p} X & & \\
 \downarrow T \times 1_U & \searrow h & \\
 \hat{X} \times U \xrightarrow{\hat{p}} \hat{X} & & Y
 \end{array}
 \quad \text{and} \quad T(x_0) = \hat{x}_0.$$

Observe that the existence of a morphism $\Sigma \rightarrow \hat{\Sigma}$ implies that $f_\Sigma = f_{\hat{\Sigma}}$.

3. COSYSTEMS

Definition 3.1. A polynomial cosystem Ξ over k is a quadruple $(A, \alpha, \beta, \gamma)$ where A is a k -algebra and $\alpha: A(Y) \rightarrow A$, $\beta: A \rightarrow A \otimes A(U)$, $\gamma: A \rightarrow k$ are k -algebra homomorphisms. If A is an affine algebra, then Ξ is an affine cosystem.

We can associate in a straightforward manner to any cosystem an input-output function mapping $A(Y)$ into $\Pi(\otimes^n A(U))$.

Definition 3.2. A morphism of cosystems $\Xi \rightarrow \hat{\Xi}$ is a k -algebra homomorphism η from A to \hat{A}

such that the following diagrams commute:

$$\begin{array}{ccc}
 A & \xrightarrow{\beta} & A \otimes A(U) \\
 \alpha \nearrow & & \downarrow \eta \otimes 1_{A(U)} \\
 A(Y) & & \\
 \hat{\alpha} \searrow & & \\
 \hat{A} & \xrightarrow{\hat{\beta}} & \hat{A} \otimes A(U)
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & & \\
 \eta \downarrow & \searrow \gamma & \\
 \hat{A} & & k \\
 \hat{\eta} \nearrow & &
 \end{array}$$

Polynomial, discrete-time, constant systems and cosystems, with their respective morphisms, form two categories which we shall call *Systems* and *Cosystems* respectively. *Affine cosystems* will be a full subcategory of *Cosystems*.

Theorem 3.3. (Duality). *Systems* and *(Affine cosystems)^{pp}* are equivalent categories.

Proof. This is a direct consequence of (1.6).

4. EQUILIBRIUM-STATE ASSUMPTION

Let $X_t = g_t(U^t)$. It is the *reachable set* at time t .

Definition 4.1. The state $x_0 \in X_\Sigma$ is an *equilibrium state* for the system Σ if there exists $u_0 \in U$ such that $p(x_0, u_0) = x_0$.

We shall restrict ourselves to considering systems having an initial state which is an equilibrium state. By a change of coordinates in U and X , we can take $x_0 = 0, u_0 = 0$ without loss of generality; this assumption then means that p has no constant term. In other words, we assume that the system was operating in a certain steady-state mode before we disturbed it with our new sequence of inputs. Such an assumption is satisfied for most classes of systems studied so far (linear, bilinear). From our point of view, the main value of the assumption is due to the following easy result:

Lemma 4.2. The sequence of reachable sets $\{X_t\}$ is monotonically increasing if and only if the initial state of the system is an equilibrium state.

5. REACHABILITY AND QUASI-REACHABILITY

Definition 5.1. A system Σ is *reachable* iff $X_R := \bigcup_{t \geq 0} X_t = X$. Σ is *reachable in bounded time* iff $X_r = X$ for some r .

Definition 5.2. The system Σ is *quasi-reachable* iff its state set is the closure (in the Zariski topology) of the reachable set, i.e. $X_Q := \overline{\bigcup_{t \geq 0} X_t} = X$. It is *quasi-reachable in bounded time* iff $\overline{X_r} = X$ for some r . A system Σ_Q with state-space X_Q can be naturally defined using the restrictions of p, h to X_Q .

Lemma 5.3. The sequence $\{\overline{X}_t\}$ is strict, i.e. $\overline{X}_t = \overline{X}_{t+1}$ implies $\overline{X}_{t+1} = \overline{X}_{t+2}$.

Proof. By definition, we have $X_{t+1} = p(X_t \times U)$, for all $t > 0$. The morphism p is continuous, hence $p(\overline{X}_t \times U) \subseteq \overline{p(X_t \times U)}$. Thus

$$\overline{X}_{t+1} = \overline{p(X_t \times U)} \supseteq p(\overline{X}_t \times U) = p(\overline{X}_t \times U) = p(\overline{X}_{t+1} \times U).$$

But of course $p(\overline{X}_{t+1} \times U) \supseteq p(X_{t+1} \times U) = X_{t+2}$. By (4.2), $X_{t+2} \supseteq X_{t+1}$. So $\overline{X}_{t+1} \supseteq X_{t+2} \supseteq X_{t+1}$, and therefore $\overline{X}_{t+1} = \overline{X}_{t+2}$.

Proposition 5.4. Let $\dim \Sigma = n$. Then the set of quasi-reachable states is \overline{X}_n .

Proof. Since U is irreducible by assumption, so is U^t . Since \overline{X}_t is the closure of the image of U^t under the morphism g_t (of algebraic sets), \overline{X}_t is also irreducible. We therefore have an increasing sequence $\overline{X}_0 \subseteq \overline{X}_1 \subseteq \overline{X}_2 \subseteq \dots$ of closed irreducible subsets of X . Since dimension of X is n ,

the preceding sequence cannot have more than $n + 1$ distinct sets, by (1.4). It follows therefore from (5.3) that this sequence cannot increase after step n .

Remark 5.5. We have just proved that a quasi-reachable system is quasi-reachable in bounded time. The same property does not hold, however, for reachability, as can be seen from the following example.

Let $U = X = Y = \mathbf{R}$, $p(x, u) = x + u^2 - 2u$, $h(x) = x$, $x_0 = 0$. Then $X_t = \{x \in \mathbf{R} \mid x \geq -t\}$. The system is clearly reachable, but not in bounded time.

We can nevertheless obtain the following characterization of reachable states:

Proposition 5.6. Let k be an uncountable, algebraically closed field. Then the reachable states form a constructible set if and only if they can be reached in bounded time.

Proof. Since U^t is a variety and g_t a morphism of algebraic sets, it follows immediately from (1.7) that X_t is a constructible set. If X_R is reached in bounded time, it is equal to an X_p , hence constructible.

The proof of the converse statement, based on Noether's Normalization Theorem, is too long to be given here.

6. FINITE CHARACTERIZATION OF THE INPUT-OUTPUT MAP

In this part we study questions related to the specification of polynomial systems. Specifically, we ask: *how many experiments are needed to characterize the input-output map of a system?*

Recall that if Σ is known to be linear, $U = k$, and if $\dim \Sigma \leq n$, then to determine the map f_Σ it is enough to know the values of $f_\Sigma(\omega)$ for the $2n$ inputs $(0, \dots, 0, 1, 0, \dots, 0)$ of length $2n$. To determine uniquely a polynomial of degree r , in general its values at $r + 1$ points must be known. Hence, to obtain similar results for nonlinear systems, some *a priori* estimates are needed for the degree of the polynomials involved in the definition of Σ . Using such estimates, one of the contributions of this paper is to *effectively compute* a bound on the number of input-output experiments sufficient to decide whether or not two systems have identical external behavior.

Theorem 6.1. Let $\Sigma, \hat{\Sigma}$ be systems of dimension $\leq n$. Write $f := f_\Sigma$, $\hat{f} := f_{\hat{\Sigma}}$. Then the following statements are equivalent:

- (i) $f = \hat{f}$,
- (ii) $f_{2n} = \hat{f}_{2n}$.

Proof. By definition $Y \subseteq k^p$ for some p . Hence the map $\bar{f}: \omega \mapsto f(\omega) - \hat{f}(\omega)$ is well defined. The map \bar{f} can be realized as $f_{\bar{\Sigma}}$, where $\bar{\Sigma}$ is defined as $\bar{X} := X \times \hat{X}$, $\bar{x}_0 := (x_0, \hat{x}_0)$, $\bar{p}((x, \hat{x}), u) := (p(x, u), \hat{p}(\hat{x}, u))$, and $h(x, \hat{x}) := h(x) - h(\hat{x})$. The theorem is then an immediate consequence of the following

Lemma 6.2. Let $f = f_\Sigma$, where Σ is a system of dimension r . Suppose that $f_t \equiv y \in Y$, a constant function. Then $f_t \equiv y$ for all $t > 0$.

Proof. Since $f_t = h \circ g_t$ for all t , it follows that $f_t \equiv y$ if and only if $h|_{X_t} \equiv y$. By hypothesis, $h|_{X_r} \equiv y$. Since h is continuous, it follows that $h|_{\bar{X}_r} \equiv y$. By (3.4), $\bar{X}_r = X_Q \supseteq X_r$, for all t . Therefore $h|_{X_t} \equiv y$ for all t .

Observation 6.3. In general, knowledge of the map f_t for some $t < 2n$ is not enough to identify a system of dimension n . Indeed, consider affine systems with $U := Y := k$, $X := k^n$ and $x_0 = 0$:

$$\Sigma: \begin{cases} x(t+1) = Fx(t) + gu(t) \\ y(t) = hx(t) + c \end{cases}$$

Then f_t is completely determined by c, a_1, \dots, a_t where $a_s = hF^{s-1}g$. It is well known from linear system theory that the data c, a_1, \dots, a_{2n} (but no proper subset of it) is sufficient to determine f .

Definition 6.4. *degree* $(\Sigma) \leq s$ iff there exist representations of p and h by polynomials having (ordinary) degree $\leq s$ separately in each variable $u_i, i = 1, \dots, m$ (where $U \subseteq k^m$) and having degree $\leq s$ jointly in the variables x_j .

If $\text{degree}(\Sigma) \leq s$ then $f_t: U^t \rightarrow Y$ admits a representation by polynomials of degree $\leq s^{t+1}$ in each variable (easy verification). From Proposition (1.12) and from Theorem (6.1) we obtain the

Corollary 6.5. Let $\Sigma, \hat{\Sigma}$ be systems of dimension $\leq n$ and of degree $\leq s$. Let $L \subseteq k$ be any set of $s^{2n+1} + 1$ elements. Let $L^{(n)} = \{(u_1, \dots, u_n) \in U^n \mid u_i \in L\}$. Then $f = \hat{f}$ if and only if $f|_{L^{(2n)}} = \hat{f}|_{L^{(2n)}}$. So the equality of behaviors of any two systems $\Sigma, \hat{\Sigma}$ is effectively decidable.

For the last statement above to be meaningful, we are of course assuming that the field k itself is effectively presented.

7. FINITE OBSERVABILITY

Definition 7.1. Two states x_1, x_2 are *distinguishable* if and only if there exists an input sequence $\omega \in U^*$ such that $h^\omega(x_1) \neq h^\omega(x_2)$, where $h^\omega(x) := h \circ p(x, \omega)$. A system is said to be *abstractly observable* when any pair of states are distinguishable.

Proposition 7.2. For any system Σ , there exists a finite set of input sequences $\omega_1, \omega_2, \dots, \omega_q$ such that $h^{\omega_i}(x_1) = h^{\omega_i}(x_2)$ for all $i \in \{1, \dots, q\}$, if and only if x_1 and x_2 are not distinguishable.

Proof. An input sequence ω enables us to define a morphism

$$l^\omega: X \times X \rightarrow k^p: (x_1, x_2) \mapsto h^\omega(x_1) - h^\omega(x_2).$$

If we compose this map with the p coordinate projections $k^p \rightarrow k: y \mapsto y_i, i = 1, \dots, p$, we get p morphisms $l_i^\omega: X \times X \rightarrow k$. The states x_1 and x_2 are distinguishable if and only if one of these morphisms is nonzero at (x_1, x_2) . Thus the states which are not distinguishable form exactly the algebraic subset of $X \times X$ defined as $V(\{l_i^\omega \mid \omega \in U^*, 1 \leq i \leq p\})$. It follows from the Hilbert Basis Theorem (1.1) that this can be defined as the set of common zeros of finitely many of the l_i^ω 's. It is therefore sufficient to apply the corresponding finitely many input sequences to determine whether any two states are distinguishable or not.

Corollary 7.3. Distinguishable states can be separated in bounded time.

We have thus proved that to test whether any two states are distinguishable or not it is sufficient to apply input sequences of bounded length to the system. In fact, we have even shown that it is enough to apply finitely many input sequences. But our result is just an existence statement and does not tell us how to choose these input sequences. The following theorem answers this question.

Theorem 7.4. Given a polynomial system Σ there exist two integers t and r (which can be computed effectively from Σ) such that the following property is true:

If L is a subset of U of cardinality r , then $h^\omega(x_1) = h^\omega(x_2)$ for all $\omega \in L$ implies $h^\omega(x_1) = h^\omega(x_2)$ for all ω in U^* .

Proof. Let $V_n = \{(x_1, x_2) \in X \times X \mid h_i^\omega(x_1) = h_i^\omega(x_2), \text{ for all } \omega \in U^j, j \leq n, 1 \leq i \leq p\}$. The degrees of the polynomials $h_i^\omega(x)$ in the variables x are bounded for all $\omega \in U^n$ because of their recursive definition. It follows that $V_t = V_{t+1} = \dots$ for some computable index t (Theorem (1.11)). Thus x_1 cannot be distinguished from x_2 if and only if $(x_1, x_2) \in V_t$. But $(x_1, x_2) \in V_t$ means that

$h_i^{\omega}(x_1) = h_i^{\omega}(x_2)$ as morphisms from U^j into k . Since the degrees of the polynomials representing these morphisms are bounded, the result follows from (1.12).

In other words, the computational aspects of our problem are reduced to those of Seidenberg's Theorem (1.11).

8. SYSTEM ISOMORPHISM

This section is concerned with the "uniqueness theorem for canonical realizations". This theorem, which is known to hold for several classes of systems (e.g. automata, linear systems) states that two "canonical" realizations of a given input/output map are "isomorphic". The appropriate notion of system isomorphism in the present context is that given by (2.5); a counterexample will show, however, that a naive definition of "canonical" as "reachable + (abstractly) observable" is not sufficient to give us the desired result.

Counterexample 8.1. We wish to construct two systems $\Sigma, \hat{\Sigma}$ with $f_{\Sigma} = f_{\hat{\Sigma}}$ both reachable and abstractly observable but such that Σ is not isomorphic to $\hat{\Sigma}$. The construction is based on the fact that (in the category of algebraic sets) a morphism which is bijective (one-to-one and onto) is not necessarily an isomorphism.

Let $q: V \rightarrow W$ be a morphism between varieties V, W such that q is a bijective map but V is not isomorphic to W . For instance, let

$$V: = k,$$

$$W: = \{(x, y) \in k^2 \mid x^3 - y^2 = 0\},$$

$$q: z \mapsto (z^2, z^3).$$

(It is a standard algebraic-geometric fact that $V \not\cong W$.)

Given any q, V, W as above, let $U: = V, Y: = W$ and define

$$\Sigma: = (V, x_0, pr_2, q),$$

$$\hat{\Sigma}: = (W, q(x_0), q \circ \hat{pr}_2, 1_W);$$

here $x_0 \in V$ is arbitrary while $pr_2: V \times V \rightarrow V$ and $\hat{pr}_2: W \times V \rightarrow V$ denote the projection on the second factor.

Since q induces a system morphism $\Sigma \rightarrow \hat{\Sigma}$, it follows that $f_{\Sigma} = f_{\hat{\Sigma}}$. Using that q is bijective, it is easy to verify that both systems are reachable and abstractly observable. Finally, Σ is not isomorphic to $\hat{\Sigma}$, because their state-sets V, W are not isomorphic.

The difficulty in proving the uniqueness theorem is most easily understood in terms of cosystems. In view of Theorem (1.8), the notion of reachability (rather quasi-reachability) is dualized into a notion of "observability" for cosystems. The notion of abstract observability for systems, however, does not have a nice dualization (see Remark 1.9)). Categorical problems like system-isomorphism are easier to study in algebraic categories like *Cosystems* rather than in geometrical-topological categories like *Systems*. Therefore, it is natural to require a stronger notion of system observability, corresponding to "cosystem reachability".

This stronger notion, which we shall call "algebraic" observability, can also be motivated by system-theoretic considerations. An intuitive definition of observability for a system Σ is that there should exist a *procedure* which permits the determination of an arbitrary initial state of Σ on the basis of appropriate input-output experiments (see a detailed discussion in Kalman [2, Chapter 10]). The word "procedure" is, of course, overly vague. In the classical context of linear systems it is well-known that *any* possible procedure is equivalent to a *linear* procedure. Already

when considering linear systems over *rings*, however, the definition of “procedure” becomes more delicate, and is in fact directly related to the existence of “observers” which are systems over the same rings, see Sontag [6]. It is natural in the present context to consider an *algebraic* notion of observability. In other words, we shall only allow *algebraic* operations on the input–output data. This is formalized as follows:

Definition 8.2. A system Σ is *algebraically observable* if there exist an integer s and $\bar{\omega} = (\omega_1, \dots, \omega_s) \in (U^*)^s$, such that for every *costate* $q \in A(X)$ there exists $\hat{q} \in A(Y^s)$ with

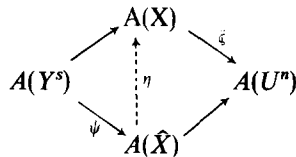
$$\hat{q} \circ H^{\bar{\omega}} = q,$$

where $H^{\bar{\omega}}: X \rightarrow Y^s$ is given by $x \mapsto (h^{\omega_1}(x), \dots, h^{\omega_s}(x))$.

Clearly this definition amounts to requiring that $A(H^{\bar{\omega}}): A(Y^s) \rightarrow A(X)$ be a surjective comorphism. The image of this comorphism is called the *observation algebra* of the system. Now the uniqueness theorem can be proved.

Theorem 8.3. Let $\Sigma, \hat{\Sigma}$ have the same input–output map. Assume that Σ is quasi-reachable and that $\hat{\Sigma}$ is algebraically observable. Then there exists a unique morphism $T: \Sigma \rightarrow \hat{\Sigma}$. If $\hat{\Sigma}$ is quasi-reachable [resp. reachable] then T is dominating [onto]. If Σ is observable then T is injective. If Σ is algebraically observable and $\hat{\Sigma}$ is quasi-reachable then T is an isomorphism.

Proof. Let the dimension of Σ be n . It follows from quasi-reachability of Σ and from (5.4) that $g_n: U^n \rightarrow X$ is dominating. So, by (1.8), $\xi = A(g_n): A(X) \rightarrow A(U^n)$ is injective. From the definition of algebraic observability we obtain $s, \bar{\omega}$ such that $\psi = A(\hat{H}^{\bar{\omega}}): A(Y^s) \rightarrow A(\hat{X})$ is surjective. Consider the diagram:



Clearly there exists a (unique) k -algebra homomorphism η making the diagram commutative. It follows from the equivalence of categories (1.6) that there exists a (unique) $T: X \rightarrow \hat{X}$ such that $A(T) = \eta$. A routine calculation shows that T gives the desired morphism.

Uniqueness of T is clear, and the rest of the proof is straightforward.

As an application, we consider the class of *state-affine* systems: $U = k^m, Y = k^p, X = k^n, h$ linear and $p(x, u)$ is of the form $p_1(u) + p_2(u)x$ (these systems are sometimes called *variable-structure* systems). In this case the maps h^{ω} are affine maps, so that the notions of abstract observability and algebraic observability coincide (see 1.9). Then:

Corollary 8.4. Let $\Sigma, \hat{\Sigma}$ be observable variable structure systems. Assume that $f_{\Sigma} = f_{\hat{\Sigma}}$. Then $\Sigma_{\mathcal{Q}}$ is isomorphic to $\hat{\Sigma}_{\mathcal{Q}}$.

CONCLUSIONS

We have presented in this paper several preliminary results relating to reachability, observability, identification, and isomorphism of polynomial systems. One of the main problems not discussed here, the problem of minimizing a given system, requires an extension of the present framework, because the category *Systems* is too “small” to permit needed operations (quotients, etc.). This amounts to saying that *Affine cosystems* is too “small” and should be replaced by *Cosystems*. By dualizing the latter, we obtain a category *Scheme Systems* containing the original category *Systems*. Recent investigation shows that a satisfactory theory of minimal realizations exists for scheme systems. Although it is not at all clear that the “physical” realization of such generalized systems would be of interest, only by embedding the theory of nonlinear systems

into such a framework can we gain the necessary perspective to analyse minimization problems. Current research efforts are directed toward this goal and toward the study of the realization problem; see Sontag [7].

Acknowledgement—We wish to thank Prof. R. E. Kalman for numerous editorial improvements in this paper.

REFERENCES

1. DIEUDONNÉ J., *Cours de Géométrie Algébrique* (2 vols), Presses Universitaires de France (1974).
2. KALMAN R. E., *Lectures on controllability and observability*, CIME Summer Course, 1968, Cremonese, Roma (1969).
3. MACDONALD I. G., *Algebraic Geometry: Introduction to Schemes*, Benjamin (1968).
4. MUMFORD, D. *Introduction to Algebraic geometry*, Harvard Lecture Notes, (1968).
5. SEIDENBERG A., On the length of a Hilbert ascending chain, *Proc. Amer. Math. Soc.* **29**, 443 (1971).
6. SONTAG E. D., Linear systems over commutative rings: a survey, *Ricerche di Automatica* (1976). To be published.
7. SONTAG E. D., *On the realization theory of polynomial input-output maps*, Ph.D. Dissertation, University of Florida (1976).