

Parametric stabilization is easy

Eduardo D. SONTAG *

Department of Mathematics, Rutgers University, New Brunswick,
NJ 08903, USA

Received 25 October 1983

Revised 8 February 1984

A polynomially parametrized family of continuous-time controllable linear systems is always stabilizable by polynomially parametrized feedback.

Keywords: Families of systems, Systems over rings.

1. Results

Theorem 1. *Let (A_λ, B_λ) be a pair of matrices, all whose entries are real polynomial functions of the parameter $\lambda \in \mathbf{R}^r$. Assume that A_λ is $n \times n$, B_λ is $n \times m$, and that the pair (A_λ, B_λ) is controllable for each $\lambda \in \mathbf{R}^r$. Then, there exists an $m \times n$ matrix K_λ whose entries are also real polynomials in λ , such that, for every λ , each eigenvalue of the matrix $A + BK$ has a negative real part.*

This result will be a consequence of the following much more general fact.

Theorem 2. *Let n, m be integers, and $A = (a_{ij})$, $B = (b_{ij})$ two matrices of distinct indeterminates, of sizes $n \times n$ and $n \times m$ respectively. Then there exist:*

* *an $m \times n$ matrix $K(A, B, \gamma)$ of real polynomials in the a_{ij}, b_{ij} , and other variable γ , and*

* *(scalar) polynomials $p(A, B, \gamma)$ and $s(A, B, \gamma)$ in the variables a_{ij}, b_{ij} , and γ ,*

such that (a) when the variables a_{ij}, b_{ij} take values making (A, B) controllable, $p(A, B, \gamma)$ is nonzero, for every real γ , and (b) for any such values of the a_{ij}, b_{ij} , and for each γ , the matrix $A + B(qK)$ has all eigenvalues with real part less than $-\gamma$ whenever q is a (real) number such that $pq > s$.

We first indicate why Theorem 1 follows from Theorem 2. Let (A_λ, B_λ) be any polynomially

parametrized family. Pick $\gamma = 0$ and choose K, p, s , as in Theorem 2. Substituting the expressions of the a_{ij}, b_{ij} as polynomials in λ into the entries of K, p, s , we may assume that these are also polynomials in λ . By (a), $p(\lambda)$ has no real zeroes. It follows by the arguments on real algebraic geometry in [3] that there is a polynomial function $q(\lambda)$ such that $pq > s$ for all λ . Thus qK is a polynomially parametrized stabilizing feedback. The same argument, specializing γ at nonzero values, gives stabilization with arbitrary convergence rates, a property which is in turn equivalent to the assumed pointwise controllability. Note that, if p would happen to be bounded below by a positive real number and if s is bounded above, then the desired q could in fact be chosen to be a constant. This will not hold in general, but coincidentally, happens to be true for the most interesting example in the literature of polynomial families of systems, as will be discussed in Section 3.

The proof of Theorem 2 is extremely simple, once that one is aware of a stabilization method, for (single, not families of) linear systems, due to Bass [1], and apparently never published. (An exposition of this method is given, however, in the textbook [16].) We shall give this argument in Section 2, but first will discuss the relation between the results here and those in previous works.

There has been a large number of papers on questions related to the stabilizability of parametrized families of systems, both continuous and discrete time. See for instance [2,3,5–13,17,18] and the references there. Many of these papers provide results for stabilization of continuous, rational, differentiable, or analytic families, by feedback laws with the same degree of smoothness. Theorem 2 (in essence due to Bass) gives a very simple proof in all the above cases – just take for instance $q = (s + 1)/p$ – if one is interested only in stabilizability (with arbitrary convergence rates). Note however that, in many of the above works one obtains, much more interestingly, either a pole-shifting result or (see below) an ‘almost pole-shifting’ property. Such stronger results are often of more relevance in control design. Further, the proof

* Research supported in part by US Air Force Grant AFOSR 80-0196.

does not generalize in any obvious way to the discrete-time case (at least if A is singular), so methods like those in [9,12] may be still needed in the latter. In connection with the results in [10] and others, note also that we *do not* need to assume here that the given systems are 'ring reachable', i.e., that controllability holds as well for complex values of λ .

In a recent note [3] we established a result on stabilization of scalar ($m = 1$) polynomial families. The proof given there is totally different from the one here, and establishes a much more precise result than the stated one of stabilization with arbitrary convergence rates. Indeed, it actually shows that, for any set S of $n - 1$ complex numbers (counting multiplicities) which is symmetric with respect to the real axis, for each $\varepsilon > 0$, and for every negative real number ρ , there is a polynomial feedback such that each $A_\lambda + B_\lambda K_\lambda$ has eigenvalues placed as follows: one is real, less than ρ , and the remaining $n - 1$, are, with the same multiplicities, at distance $< \varepsilon$ from corresponding elements of S . In other words, one can (approximately) place $n - 1$ eigenvalues arbitrarily and another one at $-\infty$. This result, proved for the scalar case in [3], will be shown in Section 3 to be false in the nonscalar case. In fact, we shall give an example where not even *continuous* approximate pole-shifting (in this sense) is possible. Then, we shall work out in detail the polynomial (and the more straightforward continuous, in fact, rational) stabilizers predicted by Theorems 1 and 2 for this example. (A generalization of the result in [3] to the nonscalar case, using *dynamic* feedback, is possible, however.) In Section 4, we provide, as a corollary to the main result, an input/output stabilization theorem.

We refer the reader to the above references for motivations regarding the study of parametrized families of systems. Let us only add a reference to [15], which obtains families of linear systems as linearizations of nonlinear systems about different operating points, with polynomial families appearing in the case of systems with finite Volterra series expansions.

2. Proof of Theorem 2

Let A, B be matrices of indeterminates as in the statement of the theorem, and let γ be another

indeterminate. Let S be the ring of (real) polynomials in the a_{ij} , the b_{ij} , and γ . Pick any polynomial $w(A, \gamma)$ such that $A + wI$ has all its eigenvalues with real part larger than 0 and such that $w > \gamma$, for every possible value of γ and of the a_{ij} . Such a choice is always possible: for instance, the spectral radius of A is bounded above by any matrix norm of A , and the Euclidean-induced norm of A is in turn bounded by the square root of $\sum a_{ij}^2$; so all eigenvalues of A are in magnitude less than, say,

$$v := 1 + \sum a_{ij}^2. \quad (2.1)$$

Thus $A + vI$ has all eigenvalues with positive real part; now pick

$$w := v + \gamma^2. \quad (2.2)$$

This insures that $A + wI$ has all eigenvalues as wanted. Alternatively, one may choose $v := n + \sum \alpha_i^2$, where the α_i denote the coefficients of the characteristic polynomial of A . (Because all eigenvalues of A are less than $\max [1, \sum |\alpha_i|] < v$ in magnitude.)

Consider the (Lyapunov) linear operator

$$\begin{aligned} \mathcal{L}: S^{n \times n} &\rightarrow S^{n \times n}, \\ \mathcal{L}(X) &= (A + wI)X + X(A + wI)'. \end{aligned} \quad (2.3)$$

(Prime indicates transpose.) Let $s(A, \gamma)$ be the determinant of \mathcal{L} . Note that, for each specialization of the a_{ij} and γ , \mathcal{L} is invertible (since $A + wI$ has no purely imaginary eigenvalues), and in particular s is a polynomial with no real zeroes. Let \mathcal{N} be the 'cofactor' transformation

$$\mathcal{N}: S^{n \times n} \rightarrow S^{n \times n} \quad (2.4)$$

such that $\mathcal{N}\mathcal{L} = \mathcal{L}\mathcal{N} = sI_n$. Now let $X = \mathcal{N}(BB')$. This is a *polynomial* matrix (over S). Moreover, from the pointwise invertibility of \mathcal{L} it follows that X is a symmetric matrix. Let Z be the cofactor matrix of X ; this is again a symmetric matrix of polynomials. We may now define the desired K as

$$K := -B'Z. \quad (2.5)$$

Finally, let

$$p := 2 \det(X). \quad (2.6)$$

Assume the variables in A, B are specialized at values making the pair (A, B) controllable and let γ take any real value. Since $-(A + wI)$ is stable

and $(-A - wI, B)$ is again controllable, it follows from Lemmas 12.1 and 12.2 in [19] that the unique solution Y of

$$(A + wI)Y + Y(A + wI)' = BB' \quad (2.7)$$

is positive definite, and in particular invertible. Now, $sY = X$, and s is always nonzero, so it follows that X itself is invertible when evaluated at such (A, B, γ) . So p is not zero, as required for the first conclusion of the theorem. Now assume that q is such that $pq > s$. Then

$$(A + wI + qBK)X + X(A + wI + qBK)' = (s - pq)BB'. \quad (2.8)$$

Since $s - pq < 0$, the matrix in the right of (2.8) can be written as $-DD'$, where D is a scalar multiple of B . (Of course, D is not a polynomial matrix anymore, but we are now arguing pointwise.) It follows that the pair

$$(A + wI + qBK, D)$$

is controllable, so again using the lemmas in [19] we conclude that $(A + qBK) + wI$ is stable. Thus $A + qBK$ has all eigenvalues with real part less than $-w$, hence less than $-\gamma$, as desired.

Remark 2.9. If only *rational* stabilization is desired, one may take the inverses of \mathcal{L} and X instead of the respective cofactor matrices, as well as $s = q = 1$ and $p = 2$. Together with the choice

$$v = n + \sum \alpha_i^2$$

discussed above, this results in an *equivariant* feedback law, in the sense of [9]: if $T \in \text{GL}(n)$, then

$$K(T^{-1}AT, T^{-1}B) = K(A, B)T.$$

This construction will be useful in Section 4.

3. An example

One of the most interesting examples regarding stabilization of families of systems is the one given in [2]; see discussions on this example in [7], and related material in [17,18]. We use here a trivial variation of it, which simplifies calculations and illustrates the same points. With $r = 2$, and using 'x, y' instead of λ_1, λ_2 , the family is the one given

by the pair

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} y & y \\ -x & z - x \end{pmatrix}, \quad (3.1)$$

where $z = 1 - x^2 - y^2$. This pair is controllable for all real (and even all complex!) x, y (see [2]), and it is impossible to solve the pole-assignment problem for it. Moreover, and this is very important in connection with the construction in [3], *it is impossible to even obtain approximate pole locations* for this example. Indeed, assume that it would be possible, for even a single pair λ, ρ with $\lambda < 0$ and $\rho < 2\lambda$, to have a feedback matrix K , even continuous on x, y , with the property that, for all x, y the two eigenvalues of $A + BK$ are one at a distance less than, say, $\varepsilon = \frac{1}{2}\lambda$ from λ , and the other being less than ρ . Since the eigenvalues remain separated, the eigenvalue $\lambda(x, y)$ close to λ would depend continuously on x, y . Thus,

$$D = A + BK + \lambda(x, y)I \quad (3.2)$$

is a continuous matrix function having a one-dimensional kernel for each x, y . It follows that ker D defines a line bundle over \mathbf{R}^2 . Let $v(x, y)$ be a section of this bundle, always nonzero. Then,

$$B(Kv) = (\lambda I - A)v \quad (3.3)$$

for all x, y . Since $\lambda(x, y)$ can never be an eigenvalue of A , it follows that $B(Kv)$ is a linear combination, with continuous coefficients, of the columns of B , which is always nonzero. But the topological degree argument in [2] shows that such combinations cannot exist. We conclude that eigenvalues cannot be placed as wanted. Thus the 'almost' pole-assignment result in [3], given there for single-input systems, cannot be generalized (with static feedback) to the non-scalar case.

It becomes then of interest to carry out the calculations for this example to obtain a stabilizing feedback, and to understand why the line-bundle argument does not apply when only stabilizability is required. One should expect complex poles and/or double roots when applying Theorems 1 and 2.

We have carried out the necessary computations, using the REDUCE symbolic manipulation system as well as double-precision FORTRAN under TOPS-20. Results are displayed in the appendixes.

First we tried a straightforward application of Theorem 2, using rational feedback. With the notations in the proof, we may choose $w = 1$ (con-

stant). The determinant of \mathcal{L} is here a constant: $s = 32$. We may take

$$q = s / \det X = 2s/p.$$

Then, $pq = 2s > s$, as desired. The resulting feedback matrix (k_{ij}) (i.e., qK , in the notations used earlier) is displayed in Appendix 1, as well as the resulting closed-loop matrix (a_{ij}) ($A + BK$, using the above notations). Note the rational form of all the matrix entries. The characteristic polynomial of the closed-loop system turns out to be

$$z^2 + 4z + 5, \tag{3.4}$$

with constant coefficients, a surprise since the original system does not have constant Kronecker indices, and hence is not equivalent, over the continuous feedback group, to a constant system [5]. Note that, as expected, the roots are complex: $-1 \mp i$.

While still in the rational case, we may ask what happens if we use instead a different q of the form $st/\det X = 2st/p$, with t different from 1. Figure 1 shows a typical root-locus, for fixed x, y but varying t , obtained for the closed-loop system. Observe the branch point after which the roots become real. The picture in this figure was computed for the case $x = y = 0$; for other values the locus is similar. The branching point occurs at

$$t = (1 + \sqrt{2})/2 \approx 1.2071$$

in this case.

Conversely, we may study what happens with a fixed gain t but varying x, y ; for instance with $t = 1000$ we get roots approaching approximately -1 and -4000 for $x = y = 0$ or large x, y , complex for suitable values of x, y , and branches at

$$x = y = 0.36613 \quad \text{and} \quad x = -y = 0.36613$$

(roots at -2000).

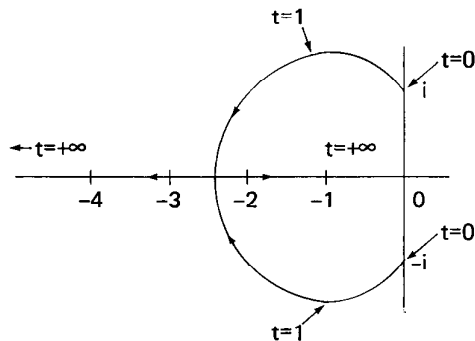


Fig. 1.

The polynomial case is even more interesting. We need to find a polynomial $q(x, y)$ such that $pq > s$ for all x, y . Consider the polynomial $p(x, y)$. Calculating, $p = 64d$, where

$$\begin{aligned} d = & x^8 + 4x^7 + 4x^6y^2 + 4x^6 + 12x^5y^2 \\ & - 4x^5 + 6x^4y^4 + 12x^4y^2 - 6x^4 + 12x^3y^4 \\ & - 8x^3y^2 + 4x^3 + 4x^2y^6 + 12x^2y^4 - 12x^2y^2 \\ & + 4x^2 + 4xy^6 - 4xy^4 + 4xy^2 - 4x + y^8 \\ & + 4y^6 - 6y^4 + 4y^2 + 1. \end{aligned}$$

Now, d is a polynomial whose highest order homogeneous component is positive definite (due to the $x^8 + y^8$ part). Since we also know that p (or d) has no real zeroes, it follows in the present case that d is bounded below by a positive constant. Thus $pq > s$ can be satisfied with a constant q . Using a Newton-Raphson method to locate the minimum of d , we get a minimum value of

$$2^{-10}10^2 = 0.09765625 \tag{3.5}$$

for d , achieved at $x = \frac{1}{2}$ and $y = 0$. Thus $pq > s$ can be satisfied for instance with $q = 1000/32$. Using this value, we computed symbolically the net feedback matrix $qK = (k_{ij})$ and closed-loop matrix (a_{ij}) , as well as the coefficients b, c of the closed-loop characteristic polynomial $z^2 + bz + c$ (see Appendix 2).

Remarkably, even though c contains large coefficients, the actual feedback law (the only part that needs to be computed by an on-line controller) has very small coefficients. The locus of closed-loop eigenvalues, as a function of x, y , has, as expected, complex cases (e.g. at $x = y = 0.37$), as well as branching points (e.g., at $x = y = 0.3599$ or $x = -y = 0.3599$).

4. An input/output interpretation

It is natural to relate the conclusion of Theorem 1 to stabilizability properties of families of transfer functions [9,13]. Let m, p be fixed positive integers. By a *polynomially* [resp., *rationaly*] *parametrized family of transfer functions* we shall mean a $p \times m$ matrix

$$W_\lambda = (p_\lambda^{ij}(z)/q_\lambda^{ij}(z))$$

such that (a) each p_λ^{ij} and q_λ^{ij} is a polynomial in z whose coefficients are polynomial [resp., rational]

functions of $\lambda \in \mathbf{R}^r$, with

$$\deg_z q_{\lambda}^{ij} > \deg_z p_{\lambda}^{ij} \quad \text{for each } i, j,$$

and (b) the leading coefficient of each q_{λ}^{ij} is independent of λ .

For instance,

$$(\lambda_1 z + \lambda_3 \lambda_2) / (z^2 + \lambda_1^2 z - (\lambda_1 + \lambda_3))$$

is such a family (with $m = p = 1$). Condition (b) is natural in the context of systems over rings; it insures that the evaluation at each λ leads to a well defined transfer function, and that the Markov parameters of W_{λ} again depend polynomially or rationally on λ .

A polynomial [resp., rational] *I/O stabilizer* for W_{λ} is given by an $m \times p$ family V_{λ} as above (note m, p are exchanged), such that the transfer function $(u_1, u_2) \rightarrow (y_1, y_2)$ in Figure 2 is stable for each λ . We then have the following:

Theorem 3. *Let W_{λ} be a rationally [resp., polynomially] parametrized family, and assume that $d(\lambda) = \text{McMillan degree of } W_{\lambda}$ is constant (as a function of λ). Then, W_{λ} admits a rational stabilizer [resp., if $r \leq 2$ there is a polynomial stabilizer].*

In the spirit of Theorem 2, one could replace the above statement by one in terms of stabilization with arbitrary convergence rates. Doing so would have the extra advantage that the condition becomes *necessary* as well (this follows by an argument using continuity of closed-loop eigenvalues), but reasons of space preclude a more detailed treatment here.

Proof of Theorem 3. In the polynomial case, note that, under the hypothesis $r \leq 2$, there exists a polynomially parametrized family $(A_{\lambda}, B_{\lambda}, C_{\lambda})$ which realizes W_{λ} , and such that $(A_{\lambda}, B_{\lambda})$ is controllable and $(A_{\lambda}, C_{\lambda})$ is observable for each λ (see

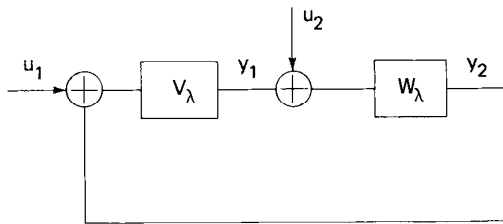


Fig. 2.

[14]). The conclusion follows then by Theorem 1 applied to $(A_{\lambda}, B_{\lambda})$ and the dual $(A'_{\lambda}, C'_{\lambda})$, together with the standard observer/state-feedback construction. In the rational case (with r arbitrary), the argument is basically that used in [9] for the analytic case (but note that the definitions of regulator are not the same as in that reference): locally in the variety of systems of degree d , a rational realization can be constructed, and this can be composed with a stabilizer construction as in Remark 2.9 through an observer/state-feedback configuration. It is easily verified that the stabilizer obtained from a given (A, B, C) is invariant under the canonical $GL(n)$ action

$$(A, B, C) \rightarrow (T^{-1}AT, T^{-1}B, CT),$$

and depends rationally on λ . Equivalence insures that the local constructions patch-up into a well defined global rational stabilizer.

References

- [1] R.W. Bass, Lecture notes on control synthesis and optimization, Report NASA Langley Res. Center (Aug. 1961).
- [2] R. Bumby, E.D. Sontag, H. Sussmann and W. Vasconcelos, Remarks on the pole-shifting problem over rings, *J. Pure Appl. Algebra* **20** (1981) 113–127.
- [3] R. Bumby and E.D. Sontag, Stabilization of polynomially parametrized families of linear systems. The single input case, *Systems and Control Letters* **3** (1983) 251–254.
- [4] R.W. Brockett, Structural properties of the equilibrium solutions of Riccati equations, in: A.V. Balakrishnan et al., Eds., *Lecture Notes in Mathematics* No. 32 (Springer, Berlin–New York, 1970) pp. 61–69.
- [5] C.I. Byrnes, On the control of certain infinite-dimensional systems by algebro-geometric techniques, *American J. Math.* **100** (1978) 1333–1381.
- [6] C.I. Byrnes, Realization theory and quadratic optimal controllers for systems defined over Banach and Fréchet algebras. *Proc. IEEE Conf. Dec. and Control* (1980) 247–255.
- [7] C.I. Byrnes, Algebraic and geometric aspects of the analysis of feedback systems, in: C.I. Byrnes and C.F. Martin, Eds., *Geometric Methods for the Theory of Linear Systems* (Reidel, Dordrecht, 1980).
- [8] D.F. Delchamps, A note on the analyticity of the Riccati metric, in: C.I. Byrnes and C.F. Martin, Eds., *Algebraic and Geometric Methods in Linear System Theory* (AMS, Providence, RI, 1980) pp. 37–42.
- [9] D.F. Delchamps, Analytic stabilization and the algebraic Riccati equation, *Proc. IEEE Conf. Dec. and Control* (1983) 1396–1401.
- [10] E. Emre and P.K. Khargonekar, Regulation of split linear systems over rings: coefficient assignment and observers, *IEEE Trans. Automat. Control* **27** (1982) 104–113.

- [11] M.L.J. Hautus and E.D. Sontag, An approach to detectability and observers, in: C.I. Byrnes and C. Martin, Eds., *AMS-SIAM Symp. Appl. Math.*, Harvard, 1979 (AMS-SIAM, 1980) pp. 99-136.
- [12] E.W. Kamen and P.K. Khargonekar, On the control of linear systems depending on parameters, *IEEE Trans. Automat. Control* **28** (1983).
- [13] P.P. Khargonekar and E.D. Sontag, On the relation between stable matrix fraction decompositions and regulable realizations of systems over rings, *IEEE Trans. Automat. Control* **27** (1982) 627-638.
- [14] Y. Rouchaleau and E.D. Sontag, On the existence of minimal realizations of linear dynamical systems over Noetherian domains, *J. Computer & System Sci.* **18** (1979) 65-75.
- [15] W.J. Rugh, Linearization about constant operating points: an input-output viewpoint, *Proc. IEEE Conf. Dec. and Control*, San Antonio (1983).
- [16] D.L. Russel, *Mathematics of Finite Dimensional Control Systems* (Marcel Dekker, New York, 1979).
- [17] A. Tannenbaum, On pole assignability over polynomial rings, *Systems and Control Letters* **2** (1982) 13-16.
- [18] A. Tannenbaum and P.K. Khargonekar, On weak pole placement of linear systems depending on parameters, submitted.
- [19] M. Wonham, *Linear Multivariable Control* (Springer, Berlin-New York, 1974).

Appendix 1. Rational feedback

$$a_{11} = p/q, \text{ where } p = 4y(x^6 + 4x^5 + 3x^4y^2 - 4x^4y + 3x^4 + 8x^3y^2 - 4x^3y - 4x^3 + 3x^2y^4 - 8x^2y^3 + 2x^2y^2 + 4x^2y - 3x^2 + 4xy^4 - 4xy^3 - 4xy^2 + 4xy) - 4y^5 - y^4 + 4y^3 + y^2 - 4y - 1, \text{ and } q = x^8 + 4x^7 + 4x^6y^2 + 4x^6 + 12x^5y^2 - 4x^5 + 6x^4y^4 + 12x^4y^2 - 6x^4 + 12x^3y^4 - 8x^3y^2 + 4x^3 + 4x^2y^6 + 12x^2y^4 - 12x^2y^2 + 4x^2 + 4xy^6 - 4xy^4 + 4xy^2 - 4x + y^8 + 4y^6 - 6y^4 + 4y^2 + 1.$$

$$a_{12} = p/q, \text{ where } p = x^8 + 4x^7 + 4x^6y^2 + 4x^6y + 4x^6 + 12x^5y^2 + 16x^5y - 4x^5 + 6x^4y^4 + 12x^4y^3 + 12x^4y^2 + 12x^3y^4 + 32x^3y^3 + 8x^3y^2 - 16x^3y + 4x^3 + 4x^2y^6 + 12x^2y^5 + 12x^2y^4 + 8x^2y^3 + 4x^2y^2 - 12x^2y + 4x^2 + 4xy^6 + 16xy^5 + 12xy^3 - 12xy^2 + 16xy - 4x + y^8 + 4y^7 + 4y^6 - 4y^5 + 10y^4 + 4y^3 + 4y^2 - 4y + 1, \text{ and } q = x^8 + 4x^7 + 4x^6y^2 + 4x^6 + 12x^5y^2 - 4x^5 + 6x^4y^4 + 12x^4y^2 - 6x^4 + 12x^3y^4 - 8x^3y^2 + 4x^3 + 4x^2y^6 + 12x^2y^4 - 12x^2y^2 + 4x^2 + 4xy^6 - 4xy^4 + 4xy^2 - 4x + y^8 + 4y^6 - 6y^4 + 4y^2 + 1.$$

$$a_{21} = p/q, \text{ where } p = -5x^8 - 20x^7 - 20x^6y^2 + 4x^6y - 20x^6 - 60x^5y^2 + 16x^5y + 20x^5 - 30x^4y^4 + 12x^4y^3 - 28x^4y^2 + 12x^4y + 30x^4 - 60x^3y^4 + 32x^3y^3 + 56x^3y^2 - 16x^3y - 20x^3 - 20x^2y^6 + 12x^2y^5 + 4x^2y^3 + 12x^2y^2 - 12x^2y - 20x^2 - 20xy^6 + 16xy^5 + 36xy^4 - 16xy^3 - 36xy^2 + 16xy + 20x - 5y^8 + 4y^7 + 12y^6 - 4y^5 - 18y^4 + 4y^3 + 12y^2 - 45, \text{ and } q = x^8 + 4x^7 + 4x^6y^2 + 4x^6 + 12x^5y^2 - 4x^5 + 6x^4y^4 + 12x^4y^2 - 6x^4 + 12x^3y^4 - 8x^3y^2 + 4x^3 + 4x^2y^6 + 12x^2y^4 - 12x^2y^2 + 4x^2 + 4xy^6 - 4xy^4 + 4xy^2 - 4x + y^8 + 4y^6 - 6y^4 + 4y^2 + 1.$$

$$a_{22} = p/q, \text{ where } p = 4(-x^8 - 4x^7 - 4x^6y^2 - x^6y - 4x^6 - 12x^5y^2 - 4x^5y + 4x^5 - 6x^4y^4 - 3x^4y^3 - 8x^4y^2 - x^4 - 12x^3y^4 - 8x^3y^3 + 12x^3y^2 + 4x^3y - 4x^3 - 4x^2y^6 - 3x^2y^5 - 4x^2y^4 - 2x^2y^3 + 8x^2y^2 + 3x^2y - 4x^2 - 4xy^6 - 4xy^5 + 8xy^4 + 4xy^3 - 8xy^2 - 4xy + 4x - y^8 - y^7 + y^4 - y^3 + y - 1), \text{ and } q = x^8 + 4x^7 + 4x^6y^2 + 4x^6 + 12x^5y^2 - 4x^5 + 6x^4y^4 + 12x^4y^2 - 6x^4 + 12x^3y^4 - 8x^3y^2 + 4x^3 + 4x^2y^6 + 12x^2y^4 - 12x^2y^2 + 4x^2 + 4xy^6 - 4xy^4 + 4xy^2 - 4x + y^8 + 4y^6 - 6y^4 + 4y^2 + 1).$$

$$k_{11} = p/q, \text{ where } p = 4(x^5 - 3x^4y + 2x^4 + 2x^3y^2 - 4x^3y - 6x^2y^3 + 4x^2y^2 + 4x^2y - 2x^2 + xy^4 - 4xy^3 + 4xy + x - 3y^5 + 2y^4 + 4y^3 - 2y^2 - 3y), \text{ and } q = x^8 + 4x^7 + 4x^6y^2 + 4x^6 + 12x^5y^2 - 4x^5 + 6x^4y^4 + 12x^4y^2 - 6x^4 + 12x^3y^4 - 8x^3y^2 + 4x^3 + 4x^2y^6 + 12x^2y^4 - 12x^2y^2 + 4x^2 + 4xy^6 - 4xy^4 + 4xy^2 - 4x + y^8 + 4y^6 - 6y^4 + 4y^2 + 1.$$

$$k_{12} = p/q, \text{ where } p = 4(x^5 - x^4y + 2x^4 + 2x^3y^2 - 2x^2y^3 + 4x^2y - 2x^2 + xy^4 + x - y^5 - 2y^4 + 4y^3 + 2y^2 - y), \text{ and } q = x^8 + 4x^7 + 4x^6y^2 + 4x^6 + 12x^5y^2 - 4x^5 + 6x^4y^4 + 12x^4y^2 - 6x^4 + 12x^3y^4 - 8x^3y^2 + 4x^3 + 4x^2y^6 + 12x^2y^4 - 12x^2y^2 + 4x^2 + 4xy^6 - 4xy^4 + 4xy^2 - 4x + y^8 + 4y^6 - 6y^4 + 4y^2 + 1.$$

$$k_{21} = p/q, \text{ where } p = 4(x^6 + 3x^5 + 3x^4y^2 - x^4y + x^4 + 6x^3y^2 - 4x^3 + 3x^2y^4 - 2x^2y^3 - 2x^2y^2 - x^2 + 3xy^4 - 4xy^2 + 3x + y^6 - y^5 - 3y^4 + 3y^2 - y - 1), \text{ and } q = x^8 + 4x^7 + 4x^6y^2 + 4x^6 + 12x^5y^2 - 4x^5 + 6x^4y^4 + 12x^4y^2 - 6x^4 + 12x^3y^4 - 8x^3y^2 + 4x^3 + 4x^2y^6 + 12x^2y^4 - 12x^2y^2 + 4x^2 + 4xy^6 - 4xy^4 + 4xy^2 - 4x + y^8 + 4y^6 - 6y^4 + 4y^2 + 1.$$

$$k_{22} = p/q, \text{ where } p = 4(x^6 + 3x^5 + 3x^4y^2 + x^4y + x^4 + 6x^3y^2 + 4x^3y - 4x^3 + 3x^2y^4 + 2x^2y^3 + 2x^2y^2 - x^2 + 3xy^4 + 4xy^3 - 4xy^2 - 4xy + 3x + y^6 + y^5 + y^4 - 1), \text{ and } q = x^8 + 4x^7 + 4x^6y^2 + 4x^6 + 12x^5y^2 - 4x^5 + 6x^4y^4 + 12x^4y^2 - 6x^4 + 12x^3y^4 - 8x^3y^2 + 4x^3 + 4x^2y^6 + 12x^2y^4 - 12x^2y^2 + 4x^2 + 4xy^6 - 4xy^4 + 4xy^2 - 4x + y^8 + 4y^6 - 6y^4 + 4y^2 + 1.$$

Appendix 2. Polynomial feedback

$$a_{11} = 125y/x^6 + 4x^5 + 3x^4y^2 - 4x^4y + 3x^4 + 8x^3y^2 - 4x^3y - 4x^3 + 3x^2y^4 - 8x^2y^3 + 2x^2y^2 + 4x^2y - 3x^2 + 4xy^4 - 4xy^3 - 4xy^2 + 4xy + 4x + y^6 - 4y^5 - y^4 + 4y^3 + y^2 - 4y - 1.$$

$$a_{12} = 125x^6y + 500x^5y + 375x^4y^3 + 375x^4y + 1000x^3y^3 + 5x^3y^2 - 500x^3y + 375x^2y^5 + 250x^2y^3 + 500x^2y^2 - 375x^2y + 500xy^5 + 500xy^4 - 500xy^3 - 500xy^2 + 500xy + 125y^7 - 125y^5 + 500y^4 + 125y^3 - 125y + 1.$$

$$a_{21} = -125x^8 - 500x^7 - 500x^6y^2 + 125x^6y - 500x^6 - 1500x^5y^2 + 500x^5y + 500x^5 - 750x^4y^4 + 375x^4y^3 - 500x^4y^2 + 375x^4y + 750x^4 - 1500x^3y^4 + 1000x^3y^3 + 1500x^3y^2 - 500x^3y - 500x^3 - 500x^2y^6 + 375x^2y^5 + 500x^2y^4 + 250x^2y^3 - 375x^2y - 500x^2 - 500xy^6 + 500xy^5 + 1000xy^4 - 500xy^3 - 1000xy^2 + 500xy + 500x - 125y^8 + 125y^7 + 500y^6 - 125y^5 - 750y^4 - 125y^3 - 125y^2 - 125y - 126.$$

$$a_{22} = 125(-x^8 - 4x^7 - 4x^6y^2 - x^6y - 4x^6 - 12x^5y^2 - 4x^5y + 4x^5 - 6x^4y^4 - 3x^4y^3 - 8x^4y^2 + 6x^4 - 12x^3y^4 - 8x^3y^3 + 12x^3y^2 + 4x^3y - 4x^3 - 4x^2y^6 - 3x^2y^5 - 4x^2y^4 - 2x^2y^3 + 8x^2y^2 + 3x^2y - 4x^2 - 4xy^6 - 4xy^5 + 8xy^4 + 4xy^3 - 8xy^2 - 4xy + 4x - y^8 - y^7 + y^3 + y - 1).$$

$$k_{11} = 125(x^5 - 3x^4y + 2x^4 + 2x^3y^2 - 4x^3y - 6x^2y^3 + 4x^2y^2 + 4x^2y - 2x^2 + xy^4 - 4xy^3 + 4xy + 2y^4 + 4y^3 - 2y^2 - 3y).$$

$$k_{12} = 125(x^5 - x^4y + 2x^4 + 2x^3y^2 - 2x^2y^3 + 4x^2y - 2x^2 + xy^4 + x - y^5 - 2y^4 + 4y^3 + 2y^2 - y).$$

$$k_{21} = 125(x^6 + 3x^5 + 3x^4y^2 - x^4y + x^4 + 6x^3y^2 - 4x^3 + 3x^2y^4 - 2x^2y^3 - 2x^2y^2 - x^2 + 3xy^4 - 4xy^2 + 3x + y^6 - y^5 - 3y^4 + 3y^2 - y - 1).$$

$$k_{22} = 125(x^6 + 3x^5 + 3x^4y^2 + x^4y + x^4 + 6x^3y^2 + 4x^3y - 4x^3 + 3x^2y^4 + 2x^2y^3 + 2x^2y^2 - x^2 + 3xy^4 + 4xy^3 - 4xy^2 - 4xy + 3x + y^6 + y^5 + y^4 - y^2 + y - 1).$$

Characteristic polynomial is z^2+bz+c , where:

$$b = 125(x^8 + 4x^7 + 4x^6y^2 + 4x^6 + 12x^5y^2 - 4x^5 + 6x^4y^4 + 12x^4y^2 - 6x^4 + 12x^3y^4 - 8x^3y^2 + 4x^2y^6 + 12x^2y^4 - 12x^2y^2 + 4x^2 + 4xy^6 - 4xy^4 + 4xy^2 - 4x + y^8 + 4y^6 - 6y^4 + 4y^2 + 1), \text{ and}$$

$$c = 31250x^{12}y^2 + 125000x^{11}y^2 + 187500x^{10}y^4 + 62500x^9y^4 - 375000x^9y^2 + 468750x^8y^6 + 312500x^8y^4 - 406250x^8y^2 + 125x^8 + 1250000x^7y^6 - 1500000x^7y^4 + 500000x^7y^2 + 500x^7 + 625000x^6y^8 + 625000x^6y^6 - 1625000x^6y^4 + 625500x^6y^2 + 500x^6 + 1250000x^5y^8 - 2250000x^5y^6 + 1500000x^5y^4 - 498500x^5y^2 - 500x^5 + 468750x^4y^{10} + 625000x^4y^8 - 2437500x^4y^6 + 1875750x^4y^4 - 405750x^4y^2 - 750x^4 + 62500000x^3y^8 + 1500000x^3y^6 - 998500x^3y^4 + 374000x^3y^2 + 500x^3 + 187500x^2y^{12} + 312500x^2y^{10} - 1625000x^2y^8 + 1875500x^2y^6 - 813000x^2y^4 + 63000x^2 + 125000xy^{12} - 375000xy^{10} + 500000xy^8 - 499500xy^6 + 374500xy^4 - 124500xy^2 - 500x + 31250y^{14} + 62500y^{12} - 406250y^{10} + 625125y^8 - 406750y^6 + 63750y^4 + 30750y^2 + 126.$$