KALMAN'S CONTROLLABILITY RANK CONDITION: FROM LINEAR TO NONLINEAR *

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Abstract

The notion of controllability was identified by Kalman as one of the central properties determining system behavior. His simple rank condition is ubiquitous in linear systems analysis. This article presents an elementary and expository overview of the generalizations of this test to a condition for testing accessibility of discrete and continuous time nonlinear systems.

1 Introduction

The state-space approach to control systems analysis took center stage in the late 50's. Right from the beginning, it was recognized that certain nondegeneracy assumptions were needed in establishing results on optimal control. However, it was not until Kalman's work ([9], [10]) that the property of *controllability* was isolated as of interest in and of itself, as it characterizes the degrees of freedom available when attempting to control a system.

The study of controllability for linear systems, first carried out in detail by Kalman and his coworkers in [10], has spanned a great number of research directions, and Kalman's citation for the IEEE Medal of Honor in 1974 attests to this influence. Associated topics such as testing degrees of controllability, and their numerical analysis aspects, are still the subject of much research (see e.g. [12] and references there). This paper deals with the

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questions associated with testing controllability of *nonlinear* systems, both those operating in *continuous time*, that is, systems of the type

$$\dot{x}(t) = f(x(t), u(t)) \tag{CT}$$

described by differential equations, and *discrete time* systems described by difference equations

$$x^+(t) = f(x(t), u(t)) \tag{DT}$$

where the superscript "+" is used to indicate time shift ($x^+(t) = x(t+1)$). In principle, one wishes to study controllability from the origin. This is the property that for each state $x \in \mathbb{R}^n$ there be some control driving 0 to x in finite time. (The terminology "reachability" is also used for this concept.) As shown below, in order to obtain elegant general results one has to weaken the notion of controllability.

To simplify matters, it will be assumed that the states $x(t)$ belong to an Euclidean space $\mathbb{R}^n$, controls $u(t)$ take values in Euclidean space $\mathbb{R}^m$, and the dynamics function f is (real-)analytic on (x, u). Many generalizations, such as allowing x to evolve on a differentiable manifold, or letting f have less smoothness, are of great interest; however, in order to keep the discussion as elementary as possible the above assumptions are made here. (Analyticity allows stating certain results in necessary and sufficient, rather than merely sufficient, manner.) The controls $u(\cdot)$ are allowed to be arbitrary measurable essentially bounded functions. The origin is assumed to be an equilibrium state, that is

$$f(0,0) = 0 \; .$$

For controllability questions from non-equilibria related results hold, except for some minor changes in definitions. An important and last restriction is that in discrete time the system (DT) will be assumed to be *invertible*, meaning that the map

$$f(\cdot, u)$$

is a diffeomorphism for each fixed u; in other words, this map is bijective and has a nonsingular differential at each point. Imposing invertibility simplifies matters considerably, and is a natural condition for equations that arise from the sampling of continuous time systems, which is one of the main ways in which discrete time systems appear in practice.

When the system is *linear*, that is,

$$f(x,u) = Ax + Bu$$

for suitable matrices A (of size $n \times n$) and B (of size $n \times m$), controllability from the origin is equivalent to the property that the rank of the $n \times nm$ Kalman block matrix

$$\Big(B, AB, A^2B, \ldots, A^{n-1}B\Big) \tag{1}$$

must equal the dimension n of the state space. This is a useful and simple test, and much effort has been spent on trying to generalize it to nonlinear systems in various forms.

The systematic study of controllability questions for continuous time nonlinear systems was begun in the early 70's. At that time, the papers [14], [19], and [13], building on previous work ([2], [5]) on partial differential equations, gave a nonlinear analogue of the above Kalman controllability rank condition. This analogue provides only a *necessary* test, not sufficient. It becomes necessary and sufficient if one is interested instead in the *accessibility* property, a weaker form of controllability which will be discussed below and which corresponds to being able to reach from the origin a set of full dimension (not necessarily the entire space). Analogous results hold also in discrete time. However, this work did not settle the question of characterizing controllability, a question which remains open and which is the subject of a major current research effort, at least in so far as characterizations of local analogues are concerned. (One does know that local controllability can be checked in principle in terms of linear relations between the Lie brackets of the vector fields defining the system ([20]), and isolating the explicit form of these relations has been a major focus of research. It is impossible to even attempt here to give a reasonably complete list of references to this very active area of research. The reference [21] can be used as a source of further bibliography.)

This brief overview article will discuss accessibility for discrete and continuous time, as well as some results which exhibit examples where accessibility and controllability coincide. Some ultimate limitations on the possibility of effectively checking controllability will be also mentioned. For more details on accessibility at an expository level, see for instance [6], [20], or [7] in continuous time, and [8] in discrete time. These references should also be consulted for justifications of all statements given here without proof. The level of the presentation will be kept as elementary as possible, in order to explain the main ideas in very simple terms.

2 Accessibility

Let Σ be either (CT) or (DT). The *reachable set* $\mathcal{R}$ is by definition the set of states reachable from the origin, that is, the set of states of the form

$$\{\phi(t,0,0,\omega) \,|\, t \geq 0, \omega \text{ admissible control } \}$$

where $\phi(t,s,x,\omega)$ denotes the value $x(t)$ at time t of the solution of (CT) or (DT) respectively, with initial condition x at time s and control function $\omega = \omega(\cdot)$. The function ω is an arbitrary sequence in the discrete time case, and is required to be measurable essentially bounded in the continuous case. If the solution of (CT) is undefined for a certain ω, then ϕ is also undefined.

The system Σ will be said to be *accessible* (from the origin) if the reachable set $\mathcal{R}$ has a nonempty interior in $\mathbb{R}^n$.

Remark 1. Accessibility can be proved to be equivalent to the following property: the set of states reachable from the origin using positive and negative time motions is a neighborhood of the origin. This equivalence, valid under the blanket assumption of analyticity that was made earlier, is often referred to as the "positive form of Chow's lemma" and is due to Krener (see [13]) for continuous time; the difference equation version is provided in [8]. It should be pointed out that for accessibility from initial states which are not equilibria, the continuous version of the equivalence is still valid, but in discrete time this equivalence does *not* follow any more; see for instance the example in [8]. ∎

Remark 2. One may also define accessibility from arbitrary initial states (rather than just from the origin). When the initial state is not an equilibrium state, however, one must distinguish between accessibility, as defined here, and "strong accessibility" which corresponds to the requirement that there be a fixed time $T > 0$ such that the reachable set in time T, that is

$$\mathcal{R}^T(x_0) := \{\phi(T,0,x_0,\omega) \,|\, \omega \text{ admissible control } \}$$

has a nonempty interior. In the case treated here, starting at an equilibrium state, both notions can be shown to coincide (see next remark for discrete time, where it is trivial to establish). ∎

Remark 3. In discrete-time, accessibility corresponds to the requirement that the union of the images of the composed maps

$$f_k(0,\cdot) : (\mathbb{R}^m)^k \to \mathbb{R}^n \quad k \geq 0$$

cover an open subset, where we are denoting

$$f_k(x,(u_1,\ldots,u_k)) := f(f(\ldots f(f(x,u_1),u_2),\ldots,u_{k-1}),u_k)$$

for every state x and sequence of controls $u_1,\ldots u_k$. By Sard's Theorem, for each fixed k it is either the case that the map $f_k(0,\cdot)$ has at least one point where its Jacobian has rank n, or its image has measure zero. Since a countable union of negligible sets again has measure zero, accessibility implies that there must exist some k and some sequence of controls $u_1,\ldots u_k$ so that the Jacobian of $f_k(0,\cdot)$ evaluated at that input sequence,

$$f_k(0,\cdot)_*[u_1,\ldots u_k] \, ,$$

has rank n. Consequently, accessibility is equivalent to accessibility in time exactly k (cf. above Remark). Moreover, accessibility is equivalent to some such rank being full (the converse follows from the Implicit Mapping Theorem). By the chain rule for derivatives, this Jacobian condition can be restated as follows: Consider the linearization of the system (DT) along the trajectory

$$x_1 = 0, x_2 = f(x_1,u_1), x_3 = f(x_2,u_2), \ldots$$

that is, the linear time-varying system

$$x(t+1) = A_t x(t) + B_t u(t)$$

with

$$A_t = \frac{\partial}{\partial x} f[x_t,u_t] \qquad B_t = \frac{\partial}{\partial u} f[x_t,u_t] \ .$$

Then accessibility is equivalent to the existence of some sequence of controls $u_1,\ldots u_k$ for which this linearization is controllable as a linear system. By analyticity, if this holds for some sequence of controls of length k then it holds for almost every such sequence. In continuous time, the same result holds too (see for instance [17] for a proof). ∎

Under certain circumstances, accessibility is equivalent to controllability. Certainly this is the case for linear systems, as is easy to see. As another example, if the system (CT) is "symmetric" meaning that

$$f(x, -u) = -f(x, u)$$

for each x and u, then accessibility from zero is equivalent to the reachable set from the origin being a neighborhood of zero, and accessibility from every point is equivalent to the reachable set being the entire space. A weaker type of symmetry is given in [3], a condition which includes linear systems and hence elegantly generalizes the equivalence of accessibility and controllability for those. Another set of sufficient conditions for the equivalence of controllability and accessibility revolve around the concept of Poisson stability; see for instance [15], [1].

3 Rank Condition – Continuous Time

For each control value u, f_u denotes the function

$$f_u : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto f(x, u)$$

(the "vector field" determined by the control u). Given any two such vector functions f and g, one can associate the new function

$$[f, g]$$

defined by the formula

$$[f, g](x) := g_*[x]f(x) - f_*[x]g(x)$$

where in general $h_*[x]$ is used to indicate the Jacobian of the vector function h evaluated at the point x. This is called the *Lie bracket* of f and g, and it represents the infinitesimal direction that results from following f and g in positive time, followed by f and g in negative time.

The *accessibility Lie algebra* $\mathcal{L}$ associated to the system (CT) is the linear span of the set of all vector functions that can be obtained starting with the

f_u's and taking any number of Lie brackets of them and the resulting functions. For instance, if u_1, u_2, u_3, u_4 are any four control values, the function

$$[[f_{u_1}, [f_{u_2}, f_{u_3}]], [f_{u_3}, f_{u_4}]]$$

is in $\mathcal{L}$.

For a linear system

$$\dot{x} = Ax + Bu$$

the functions f_u are all affine, and the Lie brackets are again of the same form. It is easy to show that all elements of $\mathcal{L}$ are generated by the elements of the form

$$A^k B v$$

where v is some vector in $\mathbb{R}^m$, or $Ax + Bu$. Moreover, every possible vector of this form does appear as some product.

The system (CT) satisfies the *accessibility rank condition* at the origin if the set of vectors

$$\mathcal{L}(0) := \{g(0), g \in \mathcal{L}\}$$

is a vector space of dimension n. In view of the preceding discussion, for linear systems this condition is the same as the Kalman controllability rank condition. The main result is then (see for instance [7]):

Theorem. The system (CT) is accessible if and only if the accessibility rank condition holds. ∎

4 Rank Condition – Discrete Time

There is an analogue of the accessibility rank condition for discrete time systems, and this is studied next. This work was started to a great extent by the papers [4], [16]; see [8] for details.

The notation f_u is as above, and in particular f_0 is the map $f(\cdot, 0)$. Recall that in the discrete case one assumes invertibility, so that the inverse maps f_u^{-1} are well-defined and again analytic. For each $i = 1, \ldots, m$ and each $u \in \mathbb{R}^m$ let

$$X_{u,i}(x) := \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} f_u \circ f_{u+\varepsilon e_i}^{-1}(x),$$

where e_i denotes the ith coordinate vector, and more generally for all u, i and each integer $k \geq 0$ let

$$(Ad_0^k X_{u,i})(x) := \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} f_0^k \circ f_u \circ f_{u+\varepsilon e_i}^{-1} \circ f_0^{-k}(x) \; .$$

The accessibility Lie algebra is now defined in terms of iterated Lie brackets of these vector functions, and the accessibility rank condition is defined in terms of this, analogously to the continuous time case. The main fact is, then, as follows.

Theorem. The system (DT) is accessible if and only if the accessibility rank condition holds. ∎

Again, for linear (discrete time) systems, the condition reduces to the Kalman controllability test. The vectors $Ad_0^k X_{u,i}$ are in fact all of the type $A^k B u$, for vectors $u \in \mathbb{R}^m$.

Remark. If the systems would only be assumed to be smooth as opposed to analytic, the accessibility condition is only sufficient but not necessary, both in discrete and continuous time. Consider for instance the system on $\mathbb{R}^2$, with $\mathbb{R}^2$ also as control space, and equations

$$\dot{x} \;=\; u_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ \alpha(x_1) \end{pmatrix}$$

where α is the function with

$$\alpha(x) = e^{-1/x^2}$$

for $x > 0$ and $\alpha(x) \equiv 0$ for $x \leq 0$. This system is easily shown to be accessible –in fact, it is completely controllable (any state can be steered to any other state),– but the accessibility rank condition does not hold. ∎

5 Controllability to the Origin

Often one is interested not in controllability *from* the origin but in controllability *to* zero. The corresponding accessibility property is that

$$\mathcal{C} := \{x \,|\, \phi(t, 0, x, \omega) = 0 \text{ for some } t \geq 0 \text{ and some admissible control } \omega\}$$

contain an open set. This might be called "accessibility *to* the origin". It corresponds to plain accessibility (from the origin) for the time-reversed system, that is,

$$\dot{x} = -f(x, u)$$

in continuous time, or

$$x^+ = f_u^{-1}(x)$$

in discrete time. Since the accessibility Lie algebra $\mathcal{L}$ is a vector space, the *same* Lie algebra results for the time-reversed of a continuous time system, proving the equivalence of both notions in that case. The same result turns out to be true in the discrete case, though the proof is much less trivial. This is summarized then by:

Proposition. A system is accessible from 0 if and only if it is accessible to 0. ∎

Remark. The proof in the discrete case relies roughly on the following argument. Introduce a superscript $-$ to the notation for the vectors $Ad_0^k X_{u,i}$ introduced above, and use $\mathcal{L}^-$ instead of $\mathcal{L}$ for the Lie algebra generated by these. Consider also the vectors

$$(Ad_0^k X_{u,i}^+)(x) := \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} f_0^k \circ f_u^{-1} \circ f_{u+\varepsilon e_i} \circ f_0^{-k}(x) \; .$$

now with $k \leq 0$, and let $\mathcal{L}^+$ be the algebra generated by these vectors. This algebra is the same as the algebra $\mathcal{L}$ obtained for the time-reversed system. One first proves that it is also possible to generate the same Lie algebra using negative k in the definition of the vectors $Ad_0^k X_{u,i}^-$ (that is, the middle term in the definition is $f_u \circ f_{u+\varepsilon e_i}^{-1}$ rather than $f_u^{-1} \circ f_{u+\varepsilon e_i}$). Thus the only obstruction is due to the use of negative instead of positive k. But since the operator

$$Ad_0 : X \mapsto Ad_0(X), \;\; Ad_0(X)(x) := (f_0^{-1})_*[f_0(x)](X(f_0(x)))$$

on the Lie algebra of all vector fields preserves the tangent space at 0, because the origin is an equilibrium state, this induces an isomorphism between the two linear subspaces $\mathcal{L}^+(0)$ and $\mathcal{L}^-(0)$, giving the desired equality of ranks. ∎

6 An Example

The following is a well-known ("folk") example from differential geometry illustrating the use of the accessibility rank condition in continuous time; because the resulting system is "symmetric" in the sense that $f(x, -u) = -f(x, u)$, and accessibility holds from every state, it can be shown that this example is completely controllable, but here we only concentrate on the local aspect about zero.

Assume that we model an automobile in the following way, as an object in the plane. The position of the center of the front axle has coordinates (x, y), its orientation is specified by the angle ϕ, and θ is the angle its wheels make relative to the orientation of the car.

We assume that the angle θ can take values on an interval $(-\theta_0, \theta_0)$, corresponding to the maximum allowed displacement of the steering wheel, and that ϕ can take arbitrary values. As controls we take the steering wheel moves (u_1) and the engine speed (u_2). Using elementary trigonometry, the following model results:

$$\dot{z} = u_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + u_2 \begin{pmatrix} \cos(\phi+\theta) \\ \sin(\phi+\theta) \\ \sin\theta \\ 0 \end{pmatrix}, \qquad (2)$$

where $z = (x, y, \phi, \theta)'$ can be thought of as belonging to the state space

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (-\theta_0, \theta_0) \subseteq \mathbb{R}^4.$$

(In fact, it is more natural to identify ϕ and $\phi + 2\pi$ and take as state space the manifold $\mathbb{R} \times \mathbb{R} \times \mathbf{S}^1 \times (-\theta_0, \theta_0)$; this leads to control systems on manifolds different from Euclidean spaces.) We take the controls as having values on $\mathbb{R}^2$; a more realistic model of course incorporates constraints on their magnitude.

A control with $u_2 \equiv 0$ corresponds to a pure steering move, while one with $u_1 \equiv 0$ models a pure driving move in which the steering wheel is fixed in one position. We let $g_1 = \mathit{steer}$ be the vector field $(0, 0, 0, 1)'$ and $g_2 = \mathit{drive}$ the vector field $(\cos(\phi+\theta), \sin(\phi+\theta), \sin\theta, 0)'$. It is intuitively clear that the system is completely controllable, but one can check accessibility using the rank condition. Indeed, computing

$$wriggle := [steer, drive]$$

and

$$slide := [wriggle, drive]$$

it is easy to see that the determinant of the matrix consisting of the columns (steer, drive, wriggle, slide) is nonzero everywhere, and in particular at the origin.

For $\phi = \theta = 0$ and any (x, y), *wriggle* is the vector $(0, 1, 1, 0)$, a mix of sliding in the y direction and a rotation, and *slide* is the vector $(0, 1, 0, 0)$ corresponding to sliding in the y direction. This means that one can in principle implement infinitesimally both of the above motions. The "wriggling" motion corresponding to *wriggle* is, from the definition of Lie bracket, basically that corresponding to many fast iterations of the actions:

steer - *drive* - *reverse steer* - *reverse drive, repeat*

which one often performs in order to get out of a tight parking space. Interestingly enough, one could also approximate the pure sliding motion: *wriggle, drive, reverse wriggle, reverse drive, repeat,* corresponding to the last vector field.

7 Remarks on Computational Complexity

It is worth looking also at Kalman's condition for linear systems from the viewpoint of a polynomial time test, in the sense of Theoretical Computer Science. One can prove that, in general, for a large class of nonlinear ("polynomial") continuous-time systems, accessibility is decidable. For a restricted class which has often appeared in applications, that of *bilinear subsystems*, accessibility can even be checked in polynomial time, just as with Kalman's test for controllability of linear systems, but the problem of true controllability is NP-hard. This last result provides a rigorous statement of the fact that accessibility is easier to characterize than controllability, and it can be interpreted as an ultimate limitation on the possibility of ever finding a characterizing condition for controllability (as opposed to accessibility) for nonlinear systems that will be as easy to check as Kalman's.

See [18] and references there for precise details on the setup as well as for proofs, as well as the more recent work [11] which extends the above.

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