

## On the Existence of Minimal Realizations of Linear Dynamical Systems over Noetherian Integral Domains\*

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This paper studies the problem of obtaining minimal realizations of linear input/output maps defined over rings. In particular, it is shown that, contrary to the case of systems over fields, it is in general impossible to obtain realizations whose dimension equals the rank of the Hankel matrix. A characterization is given of those (Noetherian) rings over which realizations of such dimensions can be always obtained, and the result is applied to delay-differential systems.

### A. INTRODUCTION

A linear, discrete-time, constant, dynamical system  $\Sigma$  over an integral domain  $R$  is defined by giving a finitely generated torsionfree  $R$ -module  $X$  (the *state module*) and a triplet of  $R$ -homomorphisms  $(F, G, H)$ , where

$$F: X \rightarrow X, \quad G: R^m \rightarrow X, \quad H: X \rightarrow R^p.$$

We call the free  $R$ -module  $R^m$  the *input module*,  $R^p$  the *output module*, and write the equations of the system

$$\begin{aligned}x_{t+1} &= Fx_t + Gu_t, & t \in Z, \\y_{t+1} &= Hx_{t+1},\end{aligned}$$

where  $u_t$  (the input at time  $t$ ) belongs to  $R^m$ ,  $x_t$  (the state at time  $t$ ) to  $X$  and  $y_t$  (the output at time  $t$ ) to  $R^p$ .

It follows from the linearity of these equations that the relation they induce between

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inputs and outputs is completely characterized by the infinite sequence of  $p \times m$   $R$ -matrices  $S = (A_1, A_2, \dots)$  (the *input/output sequence* of the system) where  $A_t$  is the matrix representation of the  $R$ -homomorphism  $HF^{t-1}G: R^m \rightarrow R^p$  in the standard bases of  $R^m$  and  $R^p$ . Conversely, given a sequence of  $p \times m$   $R$ -matrices  $S = (A_1, A_2, \dots)$ , the *realization problem* consists in finding a finitely generated torsion-free  $R$ -module  $X$  and three  $R$ -homomorphisms  $(F, G, H)$  as above such that  $A_t = HF^{t-1}G$ , for all  $t > 0$ .

Suppose that  $X$  can be generated as an  $R$ -module by  $n$  elements; then we can represent the homomorphisms  $F, G, H$  (not necessarily *uniquely*) by  $R$ -matrices with respect to the standard bases of  $R^m$  and  $R^p$  and the set of generators. We shall from now on not make any distinction between the homomorphisms and their chosen matrix representations. If  $r$  is the smallest cardinality for a set of generators of  $X$ , we shall call  $r$  the *dimension* of the system over  $R$ .

When  $R$  is a field, the realization problem is completely solved. It is shown in Kalman, Falb, and Arbib [5] that an input/output sequence can be realized by a finite-dimensional system iff its associated *behavior matrix*

$$B = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & A_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has finite rank  $n$ , that  $n$  is the minimal dimension for a realization of the sequence, and that a system realizing the sequence has minimal dimension iff it is *canonical*, i.e. both *reachable* (the map  $(G, FG, \dots, F^{n-1}G): R^{nm} \rightarrow X$  is onto) and *observable* (the map  $(H', F'H', \dots, (F')^{n-1}H')': X \rightarrow R^{np}$  is one-to-one). An algorithm is also given to construct such a minimal realization.

When  $R$  is not a field, it was first shown in Rouchaleau, Wyman, and Kalman [10] that, under fairly general conditions on  $R$ , the criterion for the existence of a realization is exactly the same as above, namely that the behavior matrix have finite rank (an up-to-date summary of these existence results is given in Section B of this paper). That paper did not consider the question of the minimal dimension of realizations. Such a concept of course is easy to define: a realizable input/output sequence  $S$  (which therefore has a behavior matrix of finite rank) can be realized by linear systems of finite dimension; a *minimal realization* of  $S$  over  $R$  will be one, the dimension of which is smaller than that of any other realization of  $S$ . It, of course, always exists but is not equivalent any more to the notion of canonical realization; its dimension may be larger than the rank of the behavior matrix, and is not in general easily determined from input/output data.

The purpose of the present paper will be to study a more restrictive and stronger version of minimality. Instead of asking, as in the aformentioned paper (Rouchaleau, Wyman, and Kalman [10]):

“When does an  $R$ -input/output sequence  $S$  realizable over the quotient field  $K$  of  $R$  also have a realization over  $R$ ?”, we shall ask:

“When does  $S$  have an  $R$ -realization which has the same dimension as a minimal realization over  $K$ ?”.

Since  $R$  is assumed to be an integral domain, we may consider its quotient field  $K$ . To a system  $(X, F, G, H)$  over  $R$  we may associate a system  $(X \otimes_R K, F \otimes_R K, G \otimes_R K,$

$H \otimes_R K$ ) over  $K$  which has the same input/output sequence of  $R$ -matrices. Furthermore, it is clear that if the system over  $R$  is canonical, then so is the associated system over  $K$ . Since the  $R$ -sequence  $S = (A_1, A_2, \dots)$  is a fortiori a  $K$ -sequence, we can find a realization for it over  $K$ ; the system  $(X \otimes_R K, F \otimes_R K, G \otimes_R K, H \otimes_R K)$  is an example of such a realization. To determine a minimal realization for the  $R$ -sequence  $S$  over  $K$  is a solved problem. We are thus led to the following

(1.1) DEFINITION. A realization  $(X, F, G, H)$  of a sequence  $S$  over  $R$  is called *absolutely minimal* iff its dimension is the same as that of a minimal realization of  $S$  over the quotient field  $K$ .

(1.2) Remark. This definition is equivalent to requesting that the system *over*  $K$  defined by the matrices  $F, G, H$  be canonical. An absolutely minimal realization remains minimal under *any* ring extension of  $R$ .

(1.3) LEMMA. *The state module  $X$  of an absolutely minimal system  $\Sigma$  is free.  $\Sigma$  is observable and weakly reachable (i.e.,  $\text{rank}_K (G, FG, \dots, F^{n-1}G) = n$ , dimension of the system), and conversely an observable and weakly reachable system is absolutely minimal.*

*Proof.* The  $n$  generators of  $X$  as an  $R$ -module are also generators of  $X \otimes_R K$  as a  $K$ -vector space. If they are not linearly independent, the dimension of  $X \otimes_R K$  is less than  $n$ , contradicting minimality over  $K$ .

If the system were not observable, there would be a state  $x \neq 0$  in  $X$  such that

$$x'(H'F'H' \dots (F')^{n-1}H') = 0;$$

but this would a fortiori mean that there is a state  $x \neq 0$  in  $X \otimes_R K$  which is unobservable for the system  $(X \otimes_R K, F, G, H)$  over  $K$ , contradicting its canonicity.

The proof of the converse is just as trivial. ■

The aim of this paper is to characterize those rings  $R$  over which any realizable input/output sequence can have an absolutely minimal realization. The interest of such a characterization is two-fold. First, it will tell us exactly when we do not lose anything (from the point of view of dimension) by realizing an input/output sequence over the ring  $R$  rather than over an overfield of  $R$ . Second, one of the motivations for studying systems over rings is their use in modeling delay-differential systems (c.f. Kamen [6]). In this case the rings under consideration are polynomial rings; a sufficient condition for the pointwise controllability of delay-differential systems is that the associated ring model be weakly reachable (see Sontag [12, Section 5]). Furthermore (as pointed out to the authors by E. Kamen) only for absolutely minimal realizations can internal stability be deduced from external (bounded input/bounded output) stability. So it is important to know the polynomial rings over which this condition is always true.

After having reviewed in Section B of this paper the conditions under which a realization exists, we shall study the problem of absolutely minimal realizations. We shall see in Section C that, for single input systems, canonical realizations are absolutely minimal over very general rings. Section D shows that for general multivariable systems this very nice

property holds only over principal-ideal domains; we then give an algorithm for constructing such a canonical, absolutely minimal realization. In Section *E*, we shall give a condition guaranteeing the existence of absolutely minimal realizations; specializing the result to the case of polynomial rings, we shall find that only those in one or two variables satisfy the condition.

## B. SURVEY OF THE CONDITIONS UNDER WHICH REALIZATIONS EXIST OVER A RING

The fact that

(2.1) THEOREM. *An input/output sequence  $S$  of matrices over a Noetherian domain  $R$  is realizable over  $R$  iff it is realizable over the quotient field  $K$  of  $R$ .*

was first established by Rouchaleau, Wyman, and Kalman [10]. To make the paper self contained, we shall now give a simple proof of this result.

It is well-known (see, for instance, Sontag [12, Lemma (3.1)]) that an input/output sequence  $S$  is realizable (whether over a field or over a ring) iff the (infinite) columns of its behavior matrix

$$B = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & A_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

may be written as linear combinations of a finite subset of columns; in other words, iff the columns of  $B$  generate a finitely generated module  $X$ . We can then obtain a canonical realization as follows: take  $X$  as state module; consider the shift operator on  $X$  defined by sending each column of  $B$  to the column occupying the same position in the next block column; it extends to a well defined module endomorphism  $F$  of  $X$  because of the Hankel pattern of  $B$ ; define a linear transformation  $G: R^m \rightarrow X$  by mapping the  $j$ th standard basis vector of  $R^m$  into the  $j$ th elementary column of  $B$ ; finally, define  $H: X \rightarrow R^p$  by taking as the image of any column of  $B$  the vector composed of the first  $p$  elements of that column (in other words, the intersection of the column with the first block row). Then  $(X, F, G, H)$  is a canonical realization of  $S$ .

Let us now assume that the  $R$ -sequence  $S$  is realizable over  $K$ , and assume that  $v_1, \dots, v_n$  are a set of basis columns for  $B$  over  $K$ . Then any column  $v$  of  $B$  can be written as

$$v = \sum_{i=1}^n \alpha_i(v) v_i, \quad \alpha_i(v) \in K.$$

The coefficients  $\alpha_i(v)$  in this linear dependence can be obtained using Cramer's formulas

$$\alpha_i(v) = \frac{\Delta_i(v)}{\Delta}, \quad \Delta_i(v), \Delta \in R.$$

Both these determinants, composed by additions and multiplications from elements of  $R$ , belong to  $R$ .

Define now  $u_i = v_i/\Delta$ . We have, for any column  $v$  of  $B$ ,

$$v = \sum_{i=1}^n \Delta_i(v) u_i, \quad \Delta_i(v) \in R,$$

hence the  $R$ -module generated by the columns  $v$  of  $B$  is contained in the  $R$ -module generated by  $u_1, \dots, u_n$ . Since “Noetherian” is equivalent to “every submodule of a finitely generated module is finitely generated”, the theorem follows.

If were lax the Noetherian assumption, then we can use the following result of Cahen and Chabert [16]:

(2.2) RESULT. *Let  $R$  be completely integrally closed and  $K$  its quotient field. Then an input/output sequence  $S$  of  $R$ -matrices is realizable over  $R$  whenever  $S$  is realizable over  $K$ . Furthermore, the monic recurrence relation of  $S$  of minimal degree over  $K$  has all of its coefficients in  $R$ .*

*Proof.* See Eilenberg [3, Chap. XVI, Theorem 12.2]. ■

But it has been shown (Rouchaleau, Kalman, and Wyman [10]) that an  $R$ -sequence is realizable over  $R$  whenever it is realizable over the integral closure  $\bar{R}$  of  $R$ . Hence we have:

(2.3) RESULT. *If the integral closure  $\bar{R}$  of a ring  $R$  is completely integrally closed, then an  $R$ -sequence is realizable over  $R$  iff it is realizable over the quotient field  $K$  of  $R$ .*

A slightly less general result was proved by Rouchaleau and Wyman [11], using a generalization of classical stability theory. Extensions to reduced rings can be found in Rouchaleau [9].

### C. SINGLE-INPUT OR SINGLE-OUTPUT SYSTEMS

Let  $S$  be such that  $m = 1$ ,  $p$  being an arbitrary finite integer. It is well known that the existence of a realization is linked to that of a monic recurrence relation between the elements of  $S$ . We have precisely.

(3.1) LEMMA. *If an input/output sequence  $S$  with  $m = 1$  satisfies a monic recurrence relation over  $R$  of degree  $n$ , then  $S$  has a realization of dimension  $n$  over  $R$ .*

*Proof.* Assume that  $\alpha_n A_k + \dots + \alpha_1 A_{k+n-1} + A_{k+n} = 0$  for all  $k > 0$ , with  $\alpha_i \in R$  for all  $i$ . Then the  $R$ -matrices

$$F = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -\alpha_n \\ 1 & 0 & 0 & \cdots & 0 & -\alpha_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -\alpha_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\alpha_2 \\ 0 & 0 & 0 & \cdots & 1 & -\alpha_1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad H = (A_1, A_2, \dots, A_n),$$

together with the state module  $R^n$  constitute a realization of size  $n$ . ■

(3.2) LEMMA. *If the domain  $R$  is integrally closed and the input/output sequence  $S$  is realizable over  $R$ , then its monic recurrence polynomial of minimal degree over the quotient field of  $R$  has all of its coefficients in  $R$ .*

*Proof.* Since  $S$  is realizable over  $R$ , it satisfies a monic recurrence relation with coefficients in  $R$  (given, for example, by the characteristic polynomial  $h(z)$  of  $F$  in one of its realizations). If we now view  $S$  as a sequence over  $K$ , the set of recurrence polynomials of  $S$  is an ideal  $J$  in  $K[z]$  (nonempty, since  $J$  contains  $h(z)$ ). This ideal is principal, hence has a monic generator  $f(z)$ , the monic recurrence polynomial of minimal degree of  $S$  over  $K$ . Thus we have:

$$h(z) = g(z)f(z), h(z) \text{ monic in } R[z], f(z) \text{ monic in } K[z].$$

This is the exact setup of Zariski and Samuel [Vol. I, Chap. V, Sect. 3, Theorem 5], and it follows from the integral closure of  $R$  that  $f(z)$  is in  $R[z]$ . ■

Assume then that our input/output sequence  $S$  has a minimal realization over  $K$  of dimension  $n$  and that  $S$  is realizable over  $R$ . This means that the associated behavior matrix  $B(S)$  has rank  $n$  (see for example Kalman, Falb, and Arbib [5, Chap. 10]), hence that the first  $n + 1$  columns of  $B(S)$  are linearly dependent. Then there is a recurrence relation of degree  $n$  over  $K$  between the elements (in the present case, vectors) of the input/output sequence. If  $R$  is integrally closed, it follows from (3.2) that, there is a monic recurrence relation of degree  $\leq n$  over  $R$ . By (3.1), there is an  $R$ -realization  $\Sigma$  of dimension  $\leq n$ . Since a realization with coefficients in  $R$  is a fortiori one with coefficients in  $K$  and the dimension of a minimal realization over  $K$  is  $n$ , this  $R$ -realization  $\Sigma$  must have dimension exactly  $n$ . Furthermore, the realization constructed in (3.1) is reachable (the columns of  $G, FG, \dots, F^{n-1}G$  generate  $R^n$ ) and observable. We have therefore proved

(3.3) PROPOSITION. *An  $R$ -realizable input/output sequence with  $m = 1$  over an integrally closed domain  $R$  has an  $R$ -realization which is both canonical and absolutely minimal.*

The argument is very similar in the case of single-output systems. The  $n$ -dimensional realization associated with the recurrence of degree  $n$  is now given by

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{pmatrix}, \quad G = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n-1} \\ A_n \end{pmatrix}, \quad H = (1 \quad 0 \quad \cdots \quad 0)$$

$R^n$  being the state module. Thus we have

(3.4) PROPOSITION. *An  $R$ -realizable input/output sequence with  $p = 1$  over an integrally closed domain has an  $R$ -realization which is absolutely minimal and observable but not necessarily reachable.*

To see that all realizations like in (3.4) are not necessarily reachable, assume that  $R$  is not a principal ideal domain and let  $\alpha, \beta \in R$  generate a nonprincipal proper ideal  $J$  of  $R$ .

Then the input/output sequence  $A_1 = (\alpha \beta) = A_2 = \dots$  is realized minimally over  $R$  by  $F = 1, G = (\alpha \beta), H = 1$ . If we take  $R$  as a state module, then this absolutely minimal realization is not reachable since  $J$  is proper. A canonical realization would have a state module isomorphic to  $J$  (which is non principal) hence its dimension would be 2, and it would not be absolutely minimal.

The dual of such a realization, on the other hand, is canonical. We have here an example of a “strongly observable” system (see Sontag [14]), as well as a breakdown of the fact that the dual of a canonical system over  $K = \text{field}$  is canonical.

(3.5) *Remark.* If we assume that  $R$  is not just integrally closed but even completely integrally closed (for example, if it is integrally closed and Noetherian) then we need assume only in (3.3) and (3.4) that the input/output sequence  $S$  has a minimal realization of dimension  $n$  over  $K$ . That it is  $R$ -realizable will follow directly from (2.2).

Let us now consider the general multivariable case.

#### D. WHEN ARE CANONICAL REALIZATIONS ALSO $R$ -MINIMAL ?

The answer to this question for multi-input, multi-output systems is very simple.

(4.1) PROPOSITION. *The canonical realization of every input/output sequence  $S$  over a Noetherian domain  $R$  is absolutely minimal if and only if  $R$  is a principal-ideal domain.*

*Proof. Sufficiency.* Let  $X$  be the state module of a canonical realization  $(X, F, G, H)$  of the sequence  $S$ ; it is by definition finitely generated and torsion free. Since  $R$  is a principal ideal domain,  $X$  is a free module. Let  $\dim X = n$ . Consider the associated  $K$ -system  $(X \otimes_R K, F \otimes_R K, G \otimes_R K, H \otimes_R K)$ . Its state space  $X \otimes_R K$  has the same dimension  $n$  (as a  $K$ -vector space) as  $X$  (as an  $R$ -module) since  $X$  is free. As was pointed out in the introduction, this associated  $K$ -system is canonical since the original  $R$ -system was. So  $n$  is the dimension of a minimal realization of  $S$  over  $K$ . The system  $(X, F, G, H)$  is therefore necessarily an absolutely minimal realization of  $S$ .

*Necessity.* Let us first establish the general fact, of interest by itself, that any finitely generated torsion-free module  $X$  may be the state module of a canonical system. Since  $X$  is finitely generated, with, say,  $m$  generators, there is a projection

$$R^m \xrightarrow{u} X \longrightarrow 0.$$

Since  $X$  is finitely generated, torsion-free, and the rings we are considering are integral domains, there is also an injection, for some  $p$ ,

$$0 \longrightarrow X \xrightarrow{v} R^p.$$

(see Rotman [8, Theorem 4.21]). So  $(X, F, G, H)$ , with  $X$  as a state module and with  $F = \text{identify}, G = u, H = v$  is a canonical system.

It was pointed out in the Introduction (Lemma 1.3) that the state module of an

absolutely minimal realization is always free. So if the state module of any canonical system is  $R$ -minimal, then any finitely generated, torsion free-module is free. But this, together with the Noetherian assumption, implies that  $R$  is a principal ideal domain. ■

We shall give an algorithm for constructing such a canonical realization. First we ascertain the rank of the behavior matrix  $B(S)$  (which can be done over any field containing  $R$ , using the rank condition of Kalman, Falb, and Arbib [5, Chap. 10, Condition 11.23]). Then we find a nonsingular submatrix  $\Phi$  of maximal rank, say  $n$ , and a basis over  $R$  for  $B_{n,n}$  (the submatrix of  $B(S)$  consisting of the first  $n$  block rows and columns of the behavior matrix). This can be done in the following way.

(i) Let  $L$  be the submatrix of  $B_{n,n}$  containing the rows of  $\Phi$ , and  $a_1$  the greatest common divisor of the elements in the first row of  $B_{n,n}$ . Call  $x_1$  the linear combination of the columns of  $L$  having  $a_1$  as its leading coefficient.

(ii) Subtract from every column  $x$  of  $L$  a multiple of  $x_1 \alpha(x)x_1$  ( $\alpha(x)$  in  $R$ ) such that the first element of  $x - \alpha(x)x_1$  be 0. This is possible by definition of  $a_1$ . We get a next matrix  $L_1$  with zero top row, and such that its columns together with  $x_1$  still generate the columns of  $L$ .

(iii) We apply the same procedure to the second row of  $L_1$ , obtaining  $x_2$  and  $L_2$ , etc... . At the end of the process, we shall have a basis made up of vectors  $(x_1, \dots, x_n)$ . Let  $\Gamma$  be the  $n \times m$  submatrix of  $L$  having its columns in the first block columns and  $\Lambda$  the  $p \times n$  submatrix of  $B_{n,n}$  corresponding to the columns of  $\Phi$  and the first block row. Then we can write out a realization (the so-called Silverman realization):

$$\begin{aligned} F &= (x_1, \dots, x_n)^{-1}(\sigma\Phi) \Phi^{-1}(x_1, \dots, x_n), \\ G &= (x_1, \dots, x_n)^{-1}\Gamma, \\ H &= \Lambda\Phi^{-1}(x_1, \dots, x_n), \end{aligned}$$

where  $\sigma$  designates the shift operator.

The matrix  $(x_1, \dots, x_n)$ , being lower triangular, is easy to invert. As to  $\Phi$ , its inverse is a byproduct of the determination of the rank of the behavior matrix.

#### E. WHEN CAN WE GUARANTEE THE EXISTENCE OF AN ABSOLUTELY MINIMAL REALIZATION?

We have just seen that we cannot expect every canonical multivariable system to be also absolutely minimal, unless the ring is a principal ideal domain. If, however, we are willing to consider absolutely minimal systems which may not be canonical, then we can guarantee their existence over more general rings. These are given exactly by the following

(5.1) THEOREM. *Every realizable input/output sequence  $S$  over a Noetherian ring  $R$  has an absolutely minimal realization iff every finitely generated reflexive module over  $R$  is free.*

*Proof. Necessity.* Let  $X$  be the state module of a canonical realization of  $S$  over  $R$  and



$M$  the state module of an absolutely minimal realization of  $S$ . By Lemma (1.3) an absolutely minimal realization is always observable, hence Zeiger's lemma (see Kalman, Falb, and Arbib [5, Chap. 10. Lemma 6.2]) implies the existence of an injection  $X \rightarrow M$ . Also, by definition of a minimal realization,  $M \otimes_R K$  is the state space of a canonical realization of  $S$  over  $K$ . So  $M \otimes_R K$  is isomorphic with  $X \otimes_R K$ . Thus we have, up to isomorphism,  $X \subset M \subset X \otimes_R K$ , where  $M$  is free (Lemma (1.3)).

We are exactly in the situation described by Bourbaki [2, Sect. 4, No. 1, Corollaire de la Proposition 1], and both  $R$ -modules  $X$  and  $M$  are "réseaux" of the  $K$ -vector space  $X \otimes_R K$ ,  $M$  being furthermore free and containing  $X$ . As is pointed out in the above reference (Proposition 3, (IV) and Remark 3),  $\text{Hom}_R(M, R) \subset \text{Hom}_R(X, R)$ , that is,  $M^* \subset X^*$ .

Now let  $X$  be a finitely generated, torsion-free module over  $R$ . Its dual  $X^*$  is finitely generated since  $R$  is Noetherian. Let  $\{u_1, \dots, u_p\}$  be generators of  $X^*$ . We may construct a canonical system  $\Sigma_0$ , with state module  $X$  as follows. We choose the number of inputs  $n$  and the matrix  $G$  as in the proof of (4.1), take for  $F$  the identity matrix and define  $H: X \rightarrow R^p$  in such a way that its  $p$  rows are given by the functions  $[u_1, \dots, u_p]$ .

Suppose the system  $\Sigma_0$  has an absolutely minimal realization  $\Sigma_n = (M, \hat{F}, \hat{G}, \hat{H})$ . By the first paragraph of the proof,  $X \subset M$ . So the map  $H: X \rightarrow R^p$  extends to a map  $\hat{H}: M \rightarrow R^p$ . Since in the system  $\Sigma_n$  the rows of  $H$  generate  $X^*$ , the generators of  $X^*$  extend to linear maps  $M \rightarrow R$ . So  $X^* \subset M^*$ . It follows that  $M^* = X^*$ . Consequently, if  $X$  is reflexive, then  $X = M$  and so  $X$  is free.

*Sufficiency.* Assume that  $(X, F, G, H)$  is a canonical system.  $X$  is therefore of finite type, hence a "réseau" of  $X \otimes_R K$  (see Bourbaki [2, Sect. 4, No. 1, Proposition 1]). It follows that  $X^*$  and  $X^{**}$  are reflexive (Bourbaki [2, Sect. 4, No. 2, comments following Théoreme 1]). So  $X$  and  $X^{**}$  have the same dual, and the map  $H: X \rightarrow R^p$  may be viewed as a map  $H^{**}: X^{**} \rightarrow R^p$ . The map  $F: X \rightarrow X$  canonically induces a map  $F^*: X^* \rightarrow X^*$ , which in turn canonically induces  $F^{**}: X^{**} \rightarrow X^{**}$ . Hence a system  $(X, F, G, H)$  canonically induces a system  $\Sigma^{**} = (X^{**}, F^{**}, G, H^{**})$  having the same input/output map. Since  $X^{**}$  is reflexive,  $\Sigma^{**}$  is free by assumption. But  $X^{**}$  is a free "réseau" of  $X \otimes_R K$  (Bourbaki [2, Sect. 4, No. 2, comments preceding Theorem 1]), hence  $\dim_R X^{**} = \dim_K(X \otimes_R K)$  (Bourbaki [2, Sect. 4, No. 1, Example 2]).  $X \otimes_R K$  being the state space of a canonical realization over  $K$ ,  $\Sigma^{**}$  is an absolutely minimal realization.

(5.2) *Remark.* Absolutely minimal realizations are not necessarily unique. In fact, they are a subclass of the lattice of *minimal-rank* realizations, studied in Sontag [13]. When  $R$  is a principal-ideal domain, this subclass coincides with the entire lattice.

We have thus obtained an abstract characterization of those rings over which absolutely minimal realizations always exist. We shall now show that *among rings of polynomials over a field only those in one or two indeterminates meet the condition of Theorem (5.1)*. (The case of one indeterminate has already been treated in the previous section).

(5.3) LEMMA. *Every finitely generated reflexive  $R$ -module is projective iff the global dimension of  $R$  is inferior or equal to 2.*

*Proof.* This result --- due to Bass --- may be found in Faith [4]. ■

(5.4) LEMMA. *If  $R$  is a ring of polynomials in two unknowns over a field, then every projective module of finite type over  $R$  is free.*

*Proof.* See Bass [1, Part II, Chap. 4, Sect. 6]. ■

Hilbert's theorem on syzygies implies that the global dimension of a polynomial ring in  $n$  unknowns is  $n$  (see for example, Kaplanski [7, Part III, Theorem 7]). This and the last two lemmas show that our claim is true.

*Observation.* A counterexample for the case of polynomials in three variables over a field  $K$  ( $R = K[x, y, z]$ ) is given by the following input/output map with  $m = 3$ ,  $p = 3$ .

$$A_1 = \begin{pmatrix} x & z & 0 \\ y & 0 & -z \\ 0 & -y & -x \end{pmatrix}, \quad A_2 = A_3 = \cdots = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Although rank  $B = 2$ , there exists no  $R$ -realization of dimension 2. Indeed, the canonical state module  $X$  is isomorphic to the column space of  $A_1$ , and this module can be proven to be reflexive but not free.

In view of (5.3), the general problem of deciding if a given  $R$  satisfies the condition of (5.1) breaks down into the subproblems: (i) determine if  $\text{global dim } R \leq 2$  (easy) and (ii) decide if finitely generated projectives over  $R$  are necessarily free. This latter problem is very difficult, but is currently an important research area in commutative algebra (viz. "Serre's conjecture", etc.).

Returning to the case of delay-differential systems mentioned in the Introduction, we deduce from (5.1), (5.2) and (5.4) that only up to two rationally independent delays may be in general allowed if realizations of the "right" dimensions are to exist.

## F. CONCLUSION

We have shown under exactly what conditions we can realize an input/output sequence over a ring with matrices over the same ring without losing any of the nice properties guaranteed by classical realization theory over a field. Admittedly, the class of rings thus characterized is rather narrow; however, it does contain, besides principal-ideal domains, polynomial rings in two indeterminates (which have applications in the theory of linear delay-differential systems studied by Kamen [6]).

It is possible to give an upper bound on the increase in size due to the choice of a canonical realization (see Swan [15]). In particular, over a Dedekind ring, it can be shown that this bound is equal to 1 (Bourbaki [2, Chap. 7, Sect. 4, No. 9, Theorem 6]). In fact, the first example of a canonical, yet nonminimal system was given to the authors by Professor B. F. Wyman using a Dedekind ring.

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## REFERENCES

1. H. BASS, "Algebraic  $K$ -Theory," Benjamin, New York, 1968.
2. N. BOURBAKI, "Algèbre Commutative," Chap. 7, Diviseurs, Hermann, Paris, 1965.
3. S. EILENBERG, "Automata, Languages, and Machines," Vol. A, Academic Press, New York, 1974.
4. C. FAITH, "Rings, Modules, and Categories," Springer-Verlag, New York/Berlin, 1973.
5. R. E. KALMAN, P. L. FALB, AND M. A. ARBIB, "Topics in Mathematical System Theory," McGraw-Hill, New York, 1969.
6. E. W. KAMEN, An algebraic theory of systems defined by convolution operators, *J. Math. System Theory* **9** (1975), 57-74.
7. I. KAPLANSKI, "Fields and Rings," Univ. of Chicago Press, Chicago, 1969.
8. J. J. ROTMAN, "Notes on Homological Algebra," Van Nostrand-Reinhold, Princeton, N.J., 1970.
9. Y. ROUCHALEAU, "Linear, discrete-time, finite-dimensional, dynamical systems over some classes of commutative rings," Ph.D. Thesis, Stanford University, 1972.
10. Y. ROUCHALEAU, B. F. WYMAN, AND R. E. KALMAN, Algebraic structure of linear dynamical systems. III. Realization theory over a commutative ring, *Proc. Nat. Acad. Sci. USA* **69** (1972), 2404-2406.
11. Y. ROUCHALEAU AND B. F. WYMAN, Linear dynamical systems over integral domains, *J. Comput. System Sci.* **9** (1974), 129-142.
12. E. D. SONTAG, Linear systems over commutative rings: A survey, *Ricerche Automatica* **7** (1976), 1-34.
13. E. D. SONTAG, The lattice of minimal realizations of response maps over rings, *Math. Systems Theory* **11** (1977), 169-175.
14. E. D. SONTAG, On split realizations of response maps over rings, *Inform. Contr.* **37** (1978), 23-33.
15. R. G. SWAN, The number of generators of a module, *Math. Z.* **102** (1967), 318-322.
16. P. J. CAHEN AND J. L. CHABERT, Éléments quasi-entiers et extensions de Fatou, *J. Algebra* **36** (1975), 185-192.