A Contraction Approach to the Hierarchical Analysis and Design of Networked Systems

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***Abstract*—This brief is concerned with the stability of continuous-time networked systems. Using contraction theory, a result is established on the network structure and the properties of the individual component subsystems and their couplings to ensure the overall contractivity of the entire network. Specifically, it is shown that a contraction property on a reduced-order matrix that quantifies the interconnection structure, coupled with contractivity/expansion estimates on the individual component subsystems, suffices to ensure that trajectories of the overall system converge towards each other.**

***Index Terms*—Contractivity.**

I. Introduction

It is often useful to break down the analysis and design of large-scale systems into two independent steps: i) at a "local" level of analysis, one imposes constraints on the structure and behavior of individual subsystems (components) that can be certified independently of any possible interconnections; ii) at a "global" level, properties of the network or interconnection graph are imposed, so as to guarantee a desired behavior for the full interconnected system provided that the subsystems satisfy the local certification requirement.

Such a "multi-scale" or hierarchical methodology is robust in the sense that a large degree of uncertainty can be tolerated in the components, only constrained by meeting appropriate specifications. There are many examples of such approaches in control theory, including among others: (1) the use of small-gain theorems to guarantee stability of a negative feedback loop provided that the components are individually stable (qualitative property of components) and the overall loop H_∞ gain is less than one, as well as nonlinear generalizations based on input to state stability [1]–[4], in which a nonlinear gain characterizes the required local properties; (2) input/output monotone systems theory [5]–[7], in which the local information is based on "input-output characteristics" (DC gains); and (3) the use of passivity-based tools [8]–[10].

In this work, we present an approach to this general principle for continuous-time systems, this time in the framework of contractive systems as defined in the sense of [11] (see also Section II). Despite being similar in spirit to the other approaches listed above, and in particular those based on the use of small-gain theorems, our results deal with incremental properties and are, in principle, easier to check for individual systems (just relying on the computation of the Jacobian).

For continuous-time systems, we show that contractivity of the overall system can be guaranteed if the matrix measures (matrix norms) of Jacobians of individual components are upper bounded and the measure (norm) of a reduced-order matrix associated to the interconnection is negative. The reduced-order matrix uses only norm and matrix-measure estimates, but no precise knowledge, of components. No assumptions are made on the networks. Directed networks, self-loops, and multiple regulatory interactions are allowed.

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II. Contraction Theory

A. Preliminaries

In this Section, we introduce some of the definitions that will be used in the rest of the technical note.

We first recall (see for instance [12]) that, given a vector norm on Euclidean space ($|\bullet|$), with its induced matrix norm $\|A\|$, the associated *matrix measure* μ is defined as the one-sided directional derivative of the matrix norm in the direction of A evaluated at the identity, that is

$$\mu(A) := \lim_{h \searrow 0} \frac{1}{h} \left(\|I + hA\| - 1 \right).$$

For example, if $|\bullet|$ is the standard Euclidean 2-norm, then $\mu(A)$ is the maximum eigenvalue of the symmetric part of A. Matrix measures are also known as "*logarithmic norms*", a concept independently introduced by Germund Dahlquist and Sergei Lozinskii in 1959, [13], [14]. The limit is known to exist, and the convergence is monotonic, see [13], [15].

B. Contraction of Continuous-Time Systems

We now consider systems of ordinary differential equations, generally time-dependent

$$\dot{x} = f(t, x) \tag{1}$$

defined for $t \in [0, \infty)$ and $x \in C \subseteq \mathbb{R}^n$. We assume that $f(t, x)$ is differentiable on x, and that $f(t, x)$, as well as the Jacobian of f with respect to x, denoted as $J(t, x) = \partial f / \partial x(t, x)$, are both continuous in (t, x). We denote by $\varphi(t, s, \xi)$ the value of the solution $x(t)$ at time t of the differential (1) with initial value $x(s) = \xi$. This solution is in principle defined only on some interval $s \leq t < s + \varepsilon$, but we will assume that $\varphi(t, s, \xi)$ is defined and belongs to C for all $t \geq s$ ("forward invariance" of the state set C).

Definition 1: We say that the continuous-time system (1) is *(infinitesimally) contracting*, with respect to a norm $|\cdot|$ in $\mathbb{R}^n$ with associated matrix measure μ, if for some constant $c > 0$, called a *contraction rate*, the following inequality holds:

$$\mu\left(J\left(x, t\right)\right) \leq -c, \quad \forall x \in C, \quad \forall t \geq 0. \tag{2}$$

The key theoretical result about continuous-time contracting systems links infinitesimal contraction to global contractivity, and is stated below. This result is well known (see for example [11]) and a self contained simple proof can be found in [16].

Theorem 1: Suppose that C is a convex subset of $\mathbb{R}^n$ and that $f(t, x)$ is (infinitesimally) contracting with contraction rate c. Then, for every two solutions $x(t) = \varphi(t, t_0, \xi)$ and $z(t) = \varphi(t, t_0, \zeta)$ of (1), it holds that

$$|x(t) - z(t)| \leq e^{-c(t - t_0)} \left|\xi - \zeta\right|, \quad \forall t \geq t_0. \tag{3}$$

A generalization to non-convex C can be obtained by using the concept of K-reachable sets introduced in [16]. In [11], a proof of Theorem

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1 is given when the norm being used in (3) is the Euclidean norm (see also [17]). The case of non Euclidean norms is also discussed in [11] and [29] and applied to the study of biological systems in [28] to paper in the Journal of Computational Biology. Historically, ideas closely related to those expressed in [11] can be traced back to [18] and even to [19] (see also [20], [21], and e.g., [17], [22] for a more exhaustive list of related references).

III. Globalization Result With Matrix Measures and Norms

We now turn to the main contributions of this note. We assume given:

1) k spaces $\mathbb{R}^{n_i}$ endowed respectively with "local" norms $|\xi_i|_{1,i}$, $i = 1,\dots,k$, and
2) an "interconnection" or "structure" norm $|x|_s$ on $\mathbb{R}^k$.

The structure norm is assumed to be *monotone*, meaning that, for any two vectors $x, y \in \mathbb{R}^k$, $0 \le x \le y \quad \Rightarrow \quad |x|_s \le |y|_s$ where an inequality such as "$x \le y$" between vectors is understood coordinate-wise, that is, $x_i \le y_i$ for all indices i. All the usual L^p norms, with $1 \le p \le \infty$ are monotone.

We let $N := n_1 + n_2 + \cdots + n_k$ and introduce a "global" norm on $\mathbb{R}^N$ as follows. Given any vector $\xi = (\xi_1^T,\dots,\xi_k^T)^t \in \mathbb{R}^N$, with $\xi_i \in \mathbb{R}^{n_i}$, $i = 1,\dots,k$

$$|\xi|_{\mathrm{G}} := \left| \left(|\xi_1|_{\mathrm{L},1}, \dots, |\xi_k|_{\mathrm{L},k} \right)^{\mathrm{T}} \right|_{\mathrm{S}}.$$

That is, the global norm is obtained by first computing the local norms $x_i = |\xi_i|_{1,i}$, and then evaluating the structure norm of the resulting vector x with components x_i. Using that the structure norm is assumed to be monotone, it is easy to show that this is indeed a norm.

For example, if all the local norms as well as the structure norms are L^p norms, with the same p, then the global norm is again the same L^p norm (on a larger space). However, more generally one may mix different norms.

Given norms $|x|_1$ and $|y|_2$ in $\mathbb{R}^q$ and $\mathbb{R}^p$ respectively, we may consider the usual induced operator norm on matrices $A \in \mathbb{R}^{p\times q}$, $\|A\|_{21} := \sup_{|x|_1=1} |Ax|_2$. In particular, for the special cases when $p = n_i$ and $q = n_j$ and the above norms, we denote this norm as $\|A\|_{1,i,j}$ defined as: $\|A\|_{1,i,j} := \sup_{|x|_{1,j}=1} |Ax|_{1,i}$. When $p = q = k$ we write: $\|A\|_s := \sup_{|x|_s=1} |Ax|_s$, and when $p = q = N$: $\|A\|_{\mathrm{G}} := \sup_{|x|_{\mathrm{G}}=1} |Ax|_{\mathrm{G}}$.

We use the notations $\mu_{\mathrm{L},i}[\cdot]$, $\mu_{\mathrm{S}}[\cdot]$, and $\mu_{\mathrm{G}}[\cdot]$ for the matrix measures (logarithmic norms) associated to $\|\cdot\|_{\mathrm{L},i,i}$, $\|\cdot\|_{\mathrm{S}}$, and $\|\cdot\|_{\mathrm{G}}$ respectively.

Given any "global" matrix $A_{\mathrm{G}} \in \mathbb{R}^{N\times N}$, we define its associated "structure" matrix $A_{\mathrm{S}} \in \mathbb{R}^{k\times k}$ as follows. We start by partitioning A_{G} in the form

$$A_{\mathrm{G}} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \dots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{pmatrix}. \tag{4}$$

Then, for each $i = 1,\dots,k$, we define the following numbers: $\widetilde{A}_{ii} := \mu_{\mathrm{L},i}[A_{ii}]$, and for each $i, j \in \{1,\dots,k\}$ with $i \ne j$, we let: $\widetilde{A}_{ij} := \|A_{ij}\|_{\mathrm{L},i,j}$. Finally, we define

$$A_{\mathrm{S}} := \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} & \dots & \widetilde{A}_{1k} \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \dots & \widetilde{A}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{A}_{k1} & \widetilde{A}_{k2} & \dots & \widetilde{A}_{kk} \end{pmatrix}.$$

Our main result is as follows:

Theorem 2: For every set of local norms on $\mathbb{R}^{n_i}$, every monotone structure norm on $\mathbb{R}^k$, and every matrix $A_{\mathrm{G}} \in \mathbb{R}^{N\times N}$, $\mu_{\mathrm{G}}[A_{\mathrm{G}}] \le \mu_{\mathrm{S}}[A_{\mathrm{S}}]$.

The proof will actually show a little more, namely that just upper bounds on the numbers $\widetilde{A}_{ij}$ could be used instead of the numbers themselves. The theorem follows from:

Lemma 1: For every set of local norms on $\mathbb{R}^{n_i}$, every monotone structure norm on $\mathbb{R}^k$, and every matrix $A_{\mathrm{G}} \in \mathbb{R}^{N\times N}$

$$\|I + hA_{\mathrm{G}}\|_{\mathrm{G}} \le \|I + hA_{\mathrm{S}}\|_{\mathrm{S}} + g(h) \quad \text{for all } h > 0 \tag{5}$$

where $g : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ is such that $g(h) = o(h)$ as $h \searrow 0$.

To see how Theorem 2 follows from Lemma 1, we recall that $\mu_{\mathrm{G}}[A_{\mathrm{G}}] = \lim_{h\searrow 0} 1/h \left(\|I + hA_{\mathrm{G}}\|_{\mathrm{G}} - 1\right)$, and similarly for $\mu_{\mathrm{S}}[A_{\mathrm{S}}]$. Subtracting unity from both sides in (5), dividing by h, and taking the limit as $h \searrow 0$, the proof of the Theorem is obtained.

We now prove the Lemma.

Proof: Pick any vector $\xi \in \mathbb{R}^N$. We will show that

$$|(I + hA_{\mathrm{G}})\xi|_{\mathrm{G}} \le \left[\|I + hA_{\mathrm{S}}\|_{\mathrm{S}} + g(h)\right] |\xi|_{\mathrm{G}} \tag{6}$$

for some function g as above. Since this holds in particular for all ξ with $|\xi|_{\mathrm{G}} = 1$, the Lemma will follow.

We need the following observation. Since, for any norm $|\cdot|$ and induced matrix norm $\|\cdot\|$, by definition, the matrix measure of a matrix B is $\mu(B) = c'(0)$, where $c(h) = \|I + hB\|$, there is a function $g(h) = o(h)$ such that $\|I + hB\| = 1 + h\mu(B) + g(h)$. In particular, there are such functions $g_i(h)$ associated to the local norms $|\cdot|_{\mathrm{L},i}$. We let $g(h) := \max\{g_1(h),\dots,g_k(h)\}$. Thus

$$\|1 + hA_{ii}\|_{\mathrm{L},i} \le 1 + h\mu_{\mathrm{L},i}[A_{ii}] + g(h), i = 1,\dots,k. \tag{7}$$

We start the proof of the Lemma by writing in block form $\xi = (\xi_1^T,\dots,\xi_k^T)^{\mathrm{T}}$, with $\xi_i \in \mathbb{R}^{n_i}$, $i = 1,\dots,k$, and denote $\eta := (I + hA_{\mathrm{G}})\xi$. In block form, $\eta = (\eta_1^T,\dots,\eta_k^T)^{\mathrm{T}}$, with $\eta_i \in \mathbb{R}^{n_i}$, $i = 1,\dots,k$, where

$$\eta_i = (1 + hA_{ii})\xi_i + \sum_{j\ne i} hA_{ij}\xi_j.$$

By definition of the global norm, we have $|\eta|_{\mathrm{G}} = |x|_{\mathrm{S}}$, where we denote $x_i := |\eta_i|_{\mathrm{L},i}$ for each $i = 1,\dots,k$ and $x = (x_1,\dots,x_k)^{\mathrm{T}}$. Similarly, $|\xi|_{\mathrm{G}} = |y|_{\mathrm{S}}$ where we denote $y_i := |\xi_i|_{\mathrm{L},i}$ for each $i = 1,\dots,k$ and $y = (y_1^T,\dots,y_k^T)^{\mathrm{T}}$.

Using the triangle inequality, we have that, for every $i = 1,\dots,k$

$$\begin{aligned} x_i &\le |(1 + hA_{ii})\xi_i|_{\mathrm{L},i} + \sum_{j\ne i} |hA_{ij}\xi_j|_{\mathrm{L},i} \\ &\le \|1 + hA_{ii}\|_{\mathrm{L},i} |\xi_i|_{\mathrm{L},i} + \sum_{j\ne i} h \|A_{ij}\|_{\mathrm{L},i,j} |\xi_j|_{\mathrm{L},j} \\ &\le [1 + h\mu_{\mathrm{L},i}[A_{ii}]] |\xi_i|_{\mathrm{L},i} + \sum_{j\ne i} h \|A_{ij}\|_{\mathrm{L},i,j} |\xi_j|_{\mathrm{L},j} \\ &\quad + g(h) |\xi_i|_{\mathrm{L},i} \\ &= z_i := \left[1 + h\widetilde{A}_{ii}\right] y_i + \sum_{j\ne i} h\widetilde{A}_{ij} y_j + g(h) y_i \end{aligned}$$

where we used the estimate (7). In terms of the following vector $z \in \mathbb{R}^k$:

$$z := (I + hA_{\mathrm{S}})y + g(h)y$$

we can summarize the above inequality as "$x \le z$" (in the coefficient-wise order), and hence, using monotonicity of the structure norm, we know that $|x|_{\mathrm{S}} \le |z|_{\mathrm{S}}$. By the triangle inequality $|x|_{\mathrm{S}} \le \|I + hA_{\mathrm{S}}\|_{\mathrm{S}} |y|_{\mathrm{S}} + g(h) |y|_{\mathrm{S}}$, recalling that $|x|_{\mathrm{S}} = |\eta|_{\mathrm{G}} = |(I + hA_{\mathrm{G}})\xi|_{\mathrm{G}}$ and $|y|_{\mathrm{S}} = |\xi|_{\mathrm{G}}$, we have that $|(I + hA_{\mathrm{G}})\xi|_{\mathrm{G}} \le \left[\|I + hA_{\mathrm{S}}\|_{\mathrm{S}} + g(h)\right] |\xi|_{\mathrm{G}}$ which is (6) and hence the Lemma is proved. ■

IV. Structured Systems

We now consider again the continuous-time system (1) and show how the globalization results can be used to prove contraction by studying a reduced order matrix obtained from the system Jacobian. Let J denote the Jacobian matrix of (1) (i.e., $J = J(t,x)$) and partition the matrix as

$$J = \begin{pmatrix} J_{11} & J_{12} & \dots & J_{1n_k} \\ J_{21} & J_{22} & \dots & J_{2n_k} \\ \vdots & \vdots & \ddots & \vdots \\ J_{n_k1} & J_{n_k2} & \dots & J_{n_kn_k} \end{pmatrix}$$

with $n_k < n$.

We then define a *structure Jacobian* as

$$J_S(t,x) = \begin{pmatrix} \tilde{J}_{11} & \tilde{J}_{12} & \dots & \tilde{J}_{1n_k} \\ \tilde{J}_{21} & \tilde{J}_{22} & \dots & \tilde{J}_{2n_k} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{J}_{n_k1} & \tilde{J}_{n_k2} & \dots & \tilde{J}_{n_kn_k} \end{pmatrix} \tag{8}$$

where for each $i = 1,\dots,n_k$: $\tilde{J}_{ii} := \mu_{L,i}(J_i(t,x))$ and for each $i,j \in \{1,\dots,k\}$ with $i \neq j$, we let $\tilde{J}_{ij} := \|J_{ij}(t,x)\|_{L,i,j}$.

The following result immediately follows from Theorem 1 and Theorem 2.

Corollary 1: Let $\mathcal{C}$ be a convex set in $\mathbb{R}^n$ and suppose that local and structure norms are picked as in Theorem 2. If there is some constant $c > 0$ such that

$$\mu_S[J_S(t,x)] \leq -c, \quad \forall x \in \mathcal{C}\ \forall t \geq 0$$

then (3) holds.

V. A Representative Application to Networked Systems

We now illustrate, using a representative example, how the multiscale approach presented in this technical note can be used to give sufficient conditions for the stability of networked systems. Specifically, we consider a network of Fitzhugh-Nagumo neuron models. (Note that the problem of deriving sufficient conditions ensuring network stability is particularly important in the context of neuronal networks as discussed, for instance, in [23], [24] and references therein.)

The key steps of our methodology can be outlined as follows: (i) we first compute the reduced-order structure matrix J_S for the network of interest [that is, we compute (8)]; (ii) we then use such reduced-order structure matrix, to design appropriate coupling protocols between the network nodes so as to guarantee the network convergence.

The network of FN systems of interest can be described as [25]

$$\begin{aligned} \dot{v}_i &= c\left(v_i + w_i - \frac{1}{3}v_i^3 + u(t)\right), i = 1,\dots,N, \\ \dot{w}_i &= -\frac{1}{c}(v_i - a + bw_i) \end{aligned} \tag{9}$$

where v_i is the membrane potential, w_i is a recovery variable, $u(t)$ is the magnitude of the stimulus current and the parameters a, b and c are assumed to be nonnegative. Clearly, in this case, the set $\mathcal{C}$ on which the network evolves is a convex set. In what follows, we will assume that $u(t)$ is periodic. We recall from [11, Sec. 3.7.vi] and [16, Theor.2] that a contracting system (evolving in a convex subset of the state space) forced by a T-periodic signal exhibits a T-periodic evolution (this property is known as entrainment [16]). The parameters are set to: $c = 6$, $a = 0$, $b = 2$. The network topology is shown in Fig. 1.

We assume the coupling protocol among nodes to be similar to the so-called excitatory-only coupling, which is believed to play an important role for the synchronization of neurons in the brain (see e.g., [26]). Specifically, we couple FN oscillators in the network on the state variable v_i via the additional coupling function

$$\tilde{h}_i(v_i,w_i) := -\gamma_1 \sum_{j\in N_i} v_j - (\gamma_2 + c)v_i, \quad \gamma_1 > 0, \gamma_2 > 0. \tag{10}$$

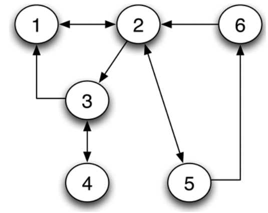

Fig. 1. Network used for the simulations of HN.

which is added to the first state equation in (9). Here N_i indicates all nodes in the neighborhood of the i-th system, i.e., directly connected to it. Thus, network dynamics becomes

$$\begin{aligned} \dot{v}_i &= c\left(v_i + w_i - \frac{1}{3}v_i^3 + u(t)\right) + \tilde{h}_i(v_i,w_i), \\ i &= 1,\dots,N, \\ \dot{w}_i &= -\frac{1}{c}(v_i - a + bw_i). \end{aligned} \tag{11}$$

To guarantee that all nodes converge towards a unique trajectory in state space, it now suffices to choose the parameters of the coupling protocol (10) so as to make the closed-loop network (11) contracting. Indeed, the synchronous subspace $\mathcal{S} := \{x_1 = \dots = x_N\}$, with $x_i = (u_i, v_i)^T$, is flow invariant for the network dynamics (that is, trajectories with initial conditions in $\mathcal{S}$ remain on it for all $t \geq t_0$). Hence, if the network is contracting, trajectories starting from any two initial conditions will converge exponentially towards each other. As trajectories in $\mathcal{S}$ remain therein for all time, it immediately follows that all trajectories must converge towards $\mathcal{S}$ and asymptotically towards each other; that is, nodes will synchronize. Moreover, network contraction also yields (see [16]) that the synchronous evolution will be periodic with the same period as $u(t)$.

Now, to study contractivity of the network dynamics we would need to study the Jacobian of (11), which is a $2N \times 2N$ matrix. Using our approach, we can look instead at the structure Jacobian, which in this case is an $N \times N$ matrix, defined as in (8). For example, let's suppose $N = 6$ and use as *local* norm on $\mathbb{R}^2$ the one induced by the ∞ vector norm to compute the structure Jacobian. Then, choosing $\gamma_2 > c + b - 1/c$, we have

$$J_S = \begin{bmatrix} -\frac{b-1}{c} & \gamma_1 & \gamma_1 & 0 & 0 & 0 \\ \gamma_1 & -\frac{b-1}{c} & 0 & 0 & \gamma_1 & \gamma_1 \\ 0 & \gamma_1 & -\frac{b-1}{c} & \gamma_1 & 0 & 0 \\ 0 & 0 & \gamma_1 & -\frac{b-1}{c} & 0 & 0 \\ 0 & \gamma_1 & 0 & 0 & -\frac{b-1}{c} & 0 \\ 0 & 0 & 0 & 0 & \gamma_1 & -\frac{b-1}{c} \end{bmatrix}.$$

Now, to ensure network contraction (and hence synchronization) we have to tune the parameter b so that a uniformly negative *structure* matrix measure $\mathbb{R}^6$ for J_S exists.

To this aim, the matrix measure induced by the 1-vector norm (on $\mathbb{R}^6$) can be used. Uniform negativity of $\mu_1(J_S)$ is obtained if: 1) $\{J_S\}_{ii} < 0$; 2) $\left|\{J_S\}_{ii}\right| > \sum_{i\neq j}\left|\{J_S\}_{ij}\right|$. Notice that the first condition above is fulfilled if $b > 1$. The second condition is instead satisfied if

$$3\gamma_1 < \frac{b-1}{c}. \tag{12}$$

Thus, if (12) is fulfilled, all network trajectories converge towards a unique synchronous solution. Fig. 2 shows a simulation for such a network, confirming the theoretical predictions.

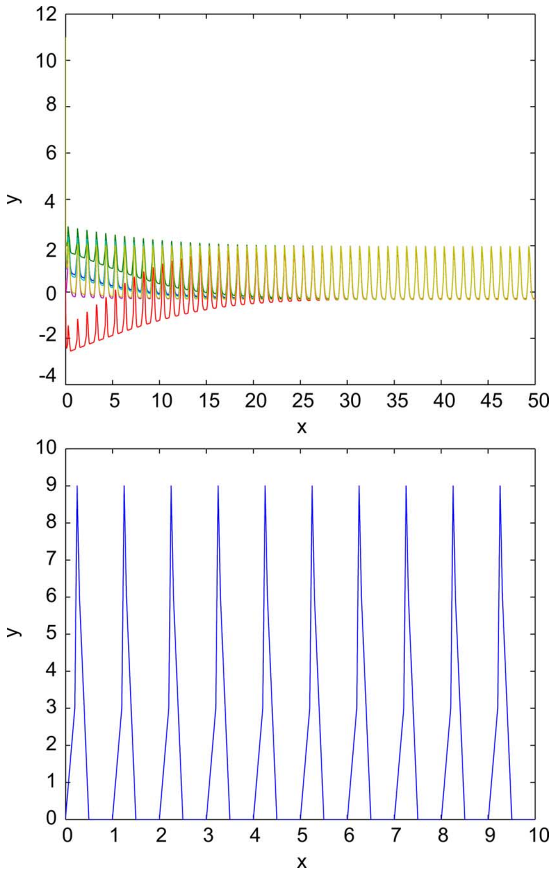

Fig. 2. Simulation of (9) with $\gamma_1 = 0.05$ and $\gamma_2 = 7$ (top panel). The behavior of $u(t)$ is shown $(0 \geq t \geq 10)$ in the bottom panel.

VI. Conclusion

We presented sufficient conditions based on contraction theory ensuring stability of networked systems. The analysis is multi-scale, and is robust in the sense that a large degree of uncertainty can be tolerated in the components, as long as the contraction estimates are met for the network subsystems and their couplings.

We view our introduction of "mixed" global norms as a key contribution. For example, as emphasized in [16], L^1 and L^∞ norms play a central role in establishing the contractivity of models of gene regulatory networks. Combining such norms with Euclidean norms for interconnections leads directly to contraction results for diffusively interconnected systems, and as a corollary, to entrainment results with respect to periodic external inputs for such interconnections.

Contraction-based analysis is a powerful tool for stability analysis, among other reasons because no *a priori* knowledge is required of an attractor in order to perform stability analysis. Moreover, the approach can be also turned into *design* tool. For example, for set-point regulation or synchronization, once a system has been designed so that a particular state or subspace is flow-invariant, contraction ensures that all system trajectories converge to this desired point or subspace.

A more general problem is that of synchronization, which means that one is not interested in contraction with respect to the global measure, but only on a weaker property, contraction to the set of states with equal coordinates. In this case, the (negative) second eigenvalue of the Laplacian matrix will determine the structure measure, so contractivity of the individual components can be relaxed. Much work remains to be done in this direction, but some preliminary results along these lines were discussed in [27].

References

[1] E. Sontag, "Input to state stability: Basic concepts and results," in *Nonlinear and Optimal Control Theory*, P. Nistri and G. Stefani, Eds. Berlin, Germany: Springer-Verlag, 2007, pp. 163–220.

[2] S. Dashkovskiy, B. Rüffer, and F. Wirth, "An ISS small-gain theorem for general networks," *Math. Control, Signals, Syst.*, vol. 19, pp. 93–122, 2007.

[3] S. Dashkovskiy, B. Rueffer, and F. Wirth, "Small gain theorems for large scale systems and construction of iss lyapunov functions," *SIAM J. Control Optim.*, vol. 48, pp. 4089–4118, 2010.

[4] S. Dashkovskiy, M. Kosmykov, and F. Wirth, "A small-gain condition for interconnections of iss systems with mixed iss characterizations," *IEEE Trans. Autom. Control*, vol. 56, no. 6, pp. 1247–1258, Jun. 2011.

[5] D. Angeli and E. Sontag, "Monotone control systems," *IEEE Trans. Autom. Control*, vol. 48, no. 10, pp. 1684–1698, 2003.

[6] E. Sontag, "Monotone and near-monotone biochemical networks," *Syst. Synthetic Biol.*, vol. 1, pp. 59–87, 2007.

[7] D. Angeli, J. E. Ferrell, and E. Sontag, "Detection of multistability, bifurcations, and hysteresis in a large class of biological positive-feedback systems," *Proc Natl Acad Sci.*, vol. 101, no. 7, pp. 1822–1827, 2004.

[8] M. Vidyasagar, *Input-Output Analysis of Large Scale Interconnected Systems*. Berlin, Germany: Springer-Verlag, 1981.

[9] P. Moylan and D. Hill, "Stability criteria for large-scale systems," *IEEE Trans. Autom. Control*, vol. AC-23, no. 2, pp. 143–149, 1978.

[10] M. Arcak and E. Sontag, "A passivity-based stability criterion for a class of interconnected systems and applications to biochemical reaction networks," *Math. Biosci. Eng.*, vol. 5, pp. 1–19, 2008.

[11] W. Lohmiller and J. J. E. Slotine, "On contraction analysis for nonlinear systems," *Automatica*, vol. 34, pp. 683–696, 1998.

[12] A. N. Michel, D. Liu, and L. Hou, *Stability of Dynamical Systems: Continuous, Discontinuous, and Discrete Systems*. New-York: Springer-Verlag, 2007.

[13] G. Dahlquist, *Stability and Error Bounds in the Numerical Integration of Ordinary Sifferential Equations*. Stockholm, Sweden: Trans. Roy. Inst. Techn, 1959.

[14] S. M. Lozinskii, "Error estimate for numerical integration of ordinary differential equations. I," *Izv. Vtssh. Uchebn. Zaved. Mat.*, vol. 5, pp. 222–222, 1959.

[15] T. Strom, "On logarithmic norms," *SIAM J. Numer. Anal.*, vol. 12, pp. 741–753, 1975.

[16] G. Russo, M. di Bernardo, and E. D. Sontag, "Global entrainment of transcriptional systems to periodic inputs," *PLoS Computational Biology*, vol. 6, p. E1000739, 2010.

[17] W. Lohmiller and J. J. Slotine, "Contraction analysis of non-linear distributed systems," *Int. J. Control*, vol. 78, pp. 678–688, 2005.

[18] P. Hartman, "On stability in the large for systems of ordinary differential equations," *Canadian J. Math.*, vol. 13, pp. 480–492, 1961.

[19] D. C. Lewis, "Metric properties of differential equations," *Amer. J. Math.*, vol. 71, pp. 294–312, 1949.

[20] A. Pavlov, A. Pogromvsky, N. v. d. Wouv, and H. Nijmeijer, "Convergent dynamics, a tribute to boris pavlovich demidovich," *Syst. Control Lett.*, vol. 52, pp. 257–261, 2004.

[21] D. Angeli, "A lyapunov approach to incremental stability properties," *IEEE Trans. Autom. Control*, vol. 47, no. 3, pp. 410–321, Mar. 2002.

[22] J. Jouffroy, "Some ancestors of contraction analysis," in *Proc. Int. Conf. Decision Control*, 2005, pp. 5450–5455.

[23] J. Peng, Z.-B. Xu, Q. Hong, and Z. Bo, "A critical analysis on global convergence of hopfield-type neural networks," *IEEE Trans. Circuits Syst.*, vol. 52, no. 4, pp. 804–814, Apr. 2005.

[24] H. Lu, "Global exponential stability analysis of cohen-grossber neural networks," *IEEE Trans. Circuits Syst. II*, vol. 52, no. 8, pp. 476–479, Aug. 2005.

[25] R. FitzHugh, "Mathematical models of threshold phenomena in the nerve membrane," *Bull. Math. Biophys.*, vol. 17, pp. 257–278, 1955.

[26] A. Winfree, "Biological rhythms and the behavior of populations of coupled oscillators," *J. Theor. Biol.*, vol. 16, pp. 15–42, 1967.

[27] G. Russo, M. di Bernardo, and E. Sontag, "Stability of networked systems: a multi-scale approach using contraction," in *Proc. IEEE Conf. Decision Control*, 2010, pp. 6559–6564.

[28] G. RussoM. di Bernardo, "How to synchronize bilogical clocks," *J. Computat. Biol.*, vol. 16, no. 2, pp. 379–393, 2009.

[29] G. RussoM. di Bernardo, "Contraction theory and master stability function: Linking two approaches to study synchronization of complex networks," *IEEE Trans. Circuits Syst. II*, vol. 56, no. 2, pp. 177–181, 2009.