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Further comments on the stabilizability of the angular velocity of a rigid body

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Abstract: We prove that the angular velocity equations can be smoothly stabilized with a single torque controller for bodies having an axis of symmetry. This complements a recent result of Aeyels and Szafranski.

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1. Introduction

In a recent paper, Aeyels and Szafranski [1] established that a rigid body can be smoothly stabilized even if just one control torque (aligned with an axis having a nonzero component along all principal axes) is allowed. Their construction results in a linear control law. Though as pointed out in that paper the feedback so obtained is highly nonrobust, it was nonetheless a rather surprising result. Their proof needs that the body have no symmetries in order for the construction to be correct. In fact, we remark below that no possible linear feedback law will work in the presence of symmetries. On the other hand, many objects, such as satellites, do typically exhibit such symmetries. In this paper we show that there exists a *non*linear control law which achieves asymptotic stabilizability.

The existence of smooth feedback controllers has been recently linked to the construction of coprime right factorizations; see [5].

To make this note more self-contained, we remind the reader of the framework in [1]. There is given a rigid body, with the components of $x = (x_1, x_2, x_3)$ denoting the angular velocity coordinates with respect to the principal axes, and the positive numbers I_1, I_2, I_3 denoting the respective principal moments of inertia. Thus the system that interests us evolves in $\mathbb{R}^3$ and has equations

$$I\dot{x} = S(x)Ix + Tu \tag{1}$$

where I is the diagonal matrix with entries I_1, I_2, I_3 and where $T = (T_1, T_2, T_3)'$ is a column vector describing the axis on which the control torque applies. We assume that

$$T_i \neq 0, \quad i = 1, 2, 3. \tag{2}$$

(See the paper [1] for a study of what may happen when some of the components of T vanish.) The controls $u(\cdot)$ are allowed to take arbitrary real values. The matrix $S(x)$ is the rotation matrix

$$S(x) = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}.$$

In [1] it is claimed that the control law $u = -T'x$ stabilizes the system (1). However, this is false if some $I_i = I_j$, $i \neq j$. (The gap is in the sentence "a simple calculation shows that..." in page 36. Actually, it is easy to prove that $I_i \neq I_j$ for all $i \neq j$ is necessary as well as sufficient for their argument to be valid.) We concentrate then on that case. Without loss of generality, we assume that $I_1 = I_2$. If $I_3 = I_1$ as well, then the equations become $\dot{x} = Tu$, which cannot be stabilized in any reasonable sense. So we assume that $I_3 \neq I_1$, i.e. that the object is not spherical.

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For the rest of this note, we assume that $I_1 = I_2$ *and* $I_3 \neq I_1$. Letting $a := (I_1 - I_3)/I_1$ and $b_i := T_i/I_i$, we obtain the final set of equations

$$\dot{x}_1 = ax_2x_3 + b_1u, \tag{3}$$

$$\dot{x}_2 = -ax_1x_3 + b_2u, \tag{4}$$

$$\dot{x}_3 = b_3u, \tag{5}$$

where the hypotheses imply that $a \neq 0$ and all $b_i \neq 0$.

Definition 1.1. Assume that $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is smooth (i.e., $\mathscr{C}^\infty$), $f(0, 0) = 0$. We shall say that the control system

$$\dot{x} = f(x, u)$$

is *(globally) smoothly stabilizable* if there exists a smooth function $k: \mathbb{R}^n \to \mathbb{R}$ such that $k(0) = 0$ and the origin is globally asymptotically stable for

$$\dot{x} = f(x, k(x)).$$

Our main result is as follows:

Theorem. *The system* (3)–(5) *is smoothly stabilizable.*

Note that it will never be possible to find a *linear* k as in [1], even if only local stabilization is required. This is because in that case there would exist states x arbitrarily close to the origin of the form

$$(x_1, x_2, 0), \quad x_1^2 + x_2^2 \neq 0,$$

for which $k(x) = 0$. Since such points are equilibria for the closed loop system, asymptotic stability cannot result. Thus truly nonlinear feedback is needed for our problem.

The next section provides the main steps of the proof of the above theorem; technical details are left for the last section. Some of the results are stated in more generality than for the particular system that we study here, and may be useful in other contexts.

2. Proof of the theorem

First of all, it is convenient to make a linear change of coordinates in (3)–(5). This will be done in two steps. First introduce variables

$$x_1' := b_3x_1 - b_1x_3, \qquad x_2' := b_3x_2 - b_2x_3.$$

Next introduce variables

$$x_1 := b_1x_1' + b_2x_2', \qquad x_2 := b_1x_2' - b_2x_1'.$$

This latter change of coordinates is invertible because its determinant is the nonzero number

$$b_1^2 + b_2^2.$$

Finally, we let $x_3 := ax_3$, $b := (b_1^2 + b_2^2)/a$, and we change coordinates in the input space by $u := ab_3u$. After all this, the equations become

$$\dot{x}_1 = x_3x_2, \tag{6}$$

$$\dot{x}_2 = -x_3x_1 - bx_3^2, \tag{7}$$

$$\dot{x}_3 = u. \tag{8}$$

This can be thought of as a cascade of a system with an integrator. To stabilize such a system, we use a lemma that has appeared often in the literature. It may be thought of as a 'generalized principle of PD control', since it asserts for mechanical systems that if one knows how to control the position of an object then one also has a feedback law for control of both position and velocity. The proof, which for completeness we repeat in the next section, can be found also in [3], as well as in [7] and [2]. The result can also be proved using the 'bounded-input bounded-output' ideas in [5]; it is only necessary to make the first system stable in that sense by a preliminary feedback [6].

Lemma 2.1. *Suppose that the following system is smoothly stabilizable*:

$$\dot{x} = f(x, u).$$

Then the system (on $\mathbb{R}^{n+1}$*) given by*

$$\dot{x} = f(x, y), \tag{9}$$

$$\dot{y} = u, \tag{10}$$

is also smoothly stabilizable. □

In order to prove the theorem, it is then necessary to verify the hypotheses of Lemma 2.1, to be applied with

$$f(x, u) = \begin{pmatrix} ux_2 \\ -ux_1 - bu^2 \end{pmatrix}. \tag{11}$$

This will be in turn a consequence of the proposition stated below.

By a *weak Lyapunov function* for

$$\dot{x} = f_0(x), \tag{12}$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, we mean a continuously differentiable scalar function $V: \mathbb{R}^n \to \mathbb{R}$ with $V(0) = 0$ and such that

$$V(x) > 0$$

(V is *positive definite*) and

$$L_{f_0}V(x) := \nabla V(x) \cdot f_0(x) \leq 0 \tag{13}$$

for each nonzero x and further, $\{x \mid V(x) \leq \delta\}$ is compact for each $\delta > 0$.

A k as in Definition 1.1 that satisfies $k(x) > 0$ for all $x \neq 0$, will be called a positive feedback. If such a k exists, we say that the system is *smoothly stabilizable with positive feedback.*

Proposition 2.2. *Assume that* f_0, $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ *are smooth,* $f_0(0) = 0$, *and there exists a weak Lyapunov function for* (12). *Suppose further that for every complete trajectory* $\{x(t), 0 \leq t < \infty\}$ *of* (12) *which satisfies*

$$L_{f_1}V(x(t)) \geq 0 \quad \textit{for all } t$$

it follows that $x(t) \equiv 0$. *Then the system*

$$\dot{x} = f_0(x) + uf_1(x) \tag{14}$$

is smoothly stabilizable with positive feedback.

This result will be proved in the next section. We now show that Lemma 2.1 and Proposition 2.2 imply our main result. To do this, we use the proposition to show that the system given by (3)–(4) satisfies the hypotheses of the lemma. We need to prove that the system (11) can be smoothly stabilized. Consider first the system obtained by dropping the factor u from these equations, which is just a forced harmonic oscillator:

$$\dot{x}_1 = x_2, \tag{15}$$

$$\dot{x}_2 = -x_1 - bu. \tag{16}$$

then $\frac{1}{2}(x_1^2 + x_2^2)$ is a weak Lyapunov function for the corresponding system with no controls, and

$$L_{f_1}V(x(t)) = bx_2(t) = b(\alpha \sin t + \beta \cos t),$$

so that $L_{f_1}V(x(t))$ can only have constant sign if $\alpha = \beta = 0$, which implies $x(t) \equiv 0$. By the proposition, there is a smooth positive feedback stabilizer $k(\cdot)$ for (15)–(16). But then

$$\dot{x}_1 = k(x)x_2,$$

$$\dot{x}_2 = -k(x)x_1 + bk(x)^2,$$

is also asymptotically stable. This latter conclusion follows from the fact that (since $k(x)$ remains nonzero) the nonzero trajectories of this system coincide with those of the system without the factor $k(x)$. (Equivalently, one may argue via Lyapunov functions.) Thus the theorem will be established once the proposition is proved.

3. Proof of the proposition

To prove the proposition we first establish the following claim:

There is a feedback stabilizer k such that $k(x) \geq 0$ for all x.

We simply define k to be any smooth function such that $k(x) \geq 0$ for all x and so that

$$k(x) > 0 \text{ if and only if } L_{f_1}V < 0.$$

(Since the set where the latter inequality holds is open, there is always some such k.) We need to see that the origin is asymptotically stable for the corresponding closed loop system. This follows by the LaSalle invariance principle: observe that

$$L_{f_0 + kf_1}V(x) = L_{f_0}V(x) + kL_{f_1}V(x) \tag{17}$$

and both terms are always nonnegative, the first by definition of weak Lyapunov function and the second by definition of k. In particular, the only way in which (17) can be zero is if both terms vanish, that is

$$L_{f_0}V(x) = 0$$

and

$$L_{f_1}V(x) \geq 0.$$

Thus there are no complete trajectories for which (17) vanishes identically, and the invariance principle applies. This establishes the claim.

We now want to perturb the feedback k in order to make it everywhere positive, as needed for the proposition. By standard inverse Lyapunov theorems (see for instance Theorem 14 in [4],) we know that there is a *strong* Lyapunov function for the system

$$\dot{x} = f_0(x) + k(x) f_1(x), \tag{18}$$

that is, there is a weak Lyapunov function which satisfies also the strict inequality

$$L_{f_0 + k f_1} V(x) < 0 \tag{19}$$

for all nonzero x. The proof will be concluded provided that we find a positive and smooth $\phi(x)$ such that

$$\alpha(x, \phi(x)) < 0 \quad \text{for all } x \neq 0 \tag{20}$$

where

$$\alpha(x, u) := L_{f_0 + k f_1 + u f_1} V$$

is a smooth function defined for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$. This is a consequence (with $W = \mathbb{R}^n$ and $\lambda = 1$) of the following more general result.

Lemma 3.1. *Let $W \subseteq \mathbb{R}^n$ be an open neighborhood of 0 and $\alpha : W \times \mathbb{R} \to \mathbb{R}$ continuous. Assume that $\alpha(x, 0) < 0$ for each $x \neq 0$. Then there exists a smooth $k : W \to \mathbb{R}$ such that $k(0) = 0$ and so that, for all nonzero x and for each $0 \leq \lambda \leq 1$, $k(x) > 0$ and $\alpha(x, \lambda k(x)) < 0$.*

Proof. Let $W_0 := W \setminus \{0\}$. We first obtain a smooth $u(\cdot) : W_0 \to \mathbb{R}$ such that

(1) $u(x) > 0$ and

(2) $\alpha(x, \lambda u(x)) < 0$,

for all $x \neq 0$ and each λ such that $0 \leq \lambda \leq 1$. Then we shall show that there exists a smooth $\phi : \mathbb{R} \to [0,1]$ such that $\phi(x) > 0$ for $x \in W_0$, $\phi(0) u(0) = 0$, and ϕu is smooth on all of W. This, together with (1) and (2) above, will imply that $k := \phi u$ is as desired.

Let $\{K_i, i \geq 1\}$ be a locally finite sequence of compact subsets of W_0 such that their interiors int K_i give a covering of W_0. Local finiteness means that

$$N_i := \{ j \mid K_j \cap K_i \neq \emptyset \}$$

is finite for all i. Since the set where $\alpha < 0$ is open and $\alpha(x, 0) < 0$ for all $x \in W_0$, there is for each i a positive number μ_i so that

$$\alpha(x, \mu) < 0$$

whenever $x \in K_i$ and $0 \leq \mu \leq \mu_i$. Pick for each i a smooth nonnegative function $u_i(\cdot)$ supported in K_i and such that

$$0 < u_i(x) \leq \frac{1}{2^i} \min\{ \mu_j, \ j \in N_i \}$$

for each $x \in$ int K_i. Finally, let

$$u := \sum_{j=1}^{\infty} u_j.$$

This is well-defined and smooth in W_0, because about each x there is only a finite number of nonzero terms. Furthermore, if $x \in K_i$ then by construction

$$0 < u(x) \leq \mu_i$$

so both properties (1) and (2) told. Now we must build ϕ as described earlier.

Fix $R > 0$ such that $\{x : \|x\| \leq R\} \subseteq W$, let N be an integer such that $NR > 1$. For $j \geq N$, let A_j be the compact annulus $\{x, 1/j \leq \|x\| \leq R\}$, and let $\theta_j : \mathbb{R} \to [0,1]$ be a smooth function such that $\theta_j(x) > 0$ if and only if $1/j < \|x\| < R$. Then the support of θ_j is precisely A_j. Let $v_j = \theta_j u$, so v_j is a compactly supported smooth function on $\mathbb{R}$. Let $\hat{K}_j$ denote the supremum, taken over all of $\mathbb{R}^n$ (or, equivalently, over A_j), of the values of the functions v_j and all their partial derivatives up to order j. Similarly, let $\bar{K}_j$ denote the supremum, taken over all of $\mathbb{R}^n$, of the values of θ_j and all its partial derivatives up to order j. Let $K_j = \max(1, \hat{K}_j, \bar{K}_j)$. Let

$$\phi_0 = \sum_{j=N}^{\infty} \frac{\theta_j}{2^j K_j}.$$

Notice that if k is any nonnegative integer then, if we look at the terms of the series for which $j \geq k$, all the derivatives of order $\leq k$ of the j-th term are bounded by 2^{-j} (because $\bar{K}_j \leq K_j$). So the series converges uniformly together with all the partial derivatives of all orders, and the limit is a smooth function on $\mathbb{R}$. Moreover, since $K_j \geq 1$, the function ϕ_0 satisfies $0 \leq \phi_0(x) \leq 1$ for all $x \in \mathbb{R}$. Clearly, $\phi_0(x) > 0$ if and only if $0 < \|x\| < R$. The function $\phi_0 u$ is given by

$$\phi_0 u = \sum_{j=N}^{\infty} \frac{\theta_j v_j}{2^j K_j}.$$

Since $\hat{K}_j \leq K_j$, we see that, if $k \geq 0$ then, if we look at the terms of the series for which $j \geq k$, then all the derivatives of order $\leq k$ of the j-th term are bounded by 2^{-j}. So the series converges uniformly together with all the partial derivatives of all orders, and the limit is a smooth function on $\mathbb{R}$. Hence $\phi_0 u$ is smooth on $\mathbb{R}$. Further, $\phi_0 u$ vanishes at the origin because each term does.

To conclude the proof, take a smooth function $\phi_1 : \mathbb{R} \to [0,1]$ such that $\phi_1(x) > 0$ whenever $\|x\| \geq R$, and ϕ_1 vanishes on a neighborhood of the origin. Then let $\phi = \frac{1}{2}(\phi_0 + \phi_1)$. It is clear that ϕ satisfies all the desired conditions. □

We conclude by providing the proof of Lemma 2.1 (cf. [3,7,2]). Assume that $\dot{x} = f(x, u)$ is smoothly stabilizable, and let

$$f_0(x) := f(x, k(x))$$

be the stable closed-loop system. Again by the Lyapunov inverse theorem, we know that there exists a positive definite function V such that

$$L_{f_0} V(x) < 0$$

for all $x \neq 0$. Since f and k are smooth, there is a smooth g defined on $\mathbb{R}^{n+1}$ so that

$$f(x, k(x) + z) = f_0(x) + zg(x, u)$$

for all x, z. Introduce the positive definite function on $\mathbb{R}^{n+1}$:

$$W(x, y) := V(x) + \tfrac{1}{2}(y - k(x))^2$$

and take the feedback

$$k'(x, y) := \nabla k(x) \cdot f(x, y) - \nabla V(x) \cdot g(x, z) - z,$$

where $z := y - k(x)$. Then, the derivative of W along trajectories of (9), (10) with $u = k'(x, y)$ is

$$L_{f_0} V(x) - z^2 < 0$$

for all nonzero (x, y), and global stability is assured.

The proof of the theorem is now complete.

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