International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

An algebraic approach to bounded controllability of linear systems†

EDUARDO D. SONTAG [a]

[a] Department of Mathematics , Rutgers University , New Brunswick, New Jersey, 08903, U.S.A.

Published online: 27 Mar 2007.

INT. J. CONTROL, 1984, VOL. 39, NO. 1, 181–188

An algebraic approach to bounded controllability of linear systems†

EDUARDO D. SONTAG‡

In this paper we present an algebraic approach to the proof that a linear system with matrices (A, B) is null-controllable using bounded inputs if and only if it is null-controllable (with unbounded inputs) and all eigenvalues of A have non-positive real parts (continuous time) or magnitude not greater than one (discrete time). We also give the analogous results for the asymptotic case. Finally, we give an interpretation of these results in the context of local non-linear controllability.

1. Introduction

Let (A, B) be a pair of real matrices satisfying the usual Kalman reachability condition : rank $(A, AB, \ldots, A^{n-1}B) = n =$ size of A. Assume that B has m columns and that the input space is now restricted to a bounded subset U of R^m which contains 0 in its interior. We wish to know if the corresponding continuous and discrete time systems (with control space U) are null-controllable (NC). Since these systems are *locally* NC, it is clear that a sufficient condition for NC to hold is for A to be a (continuous or discrete) stability matrix. It is also true, but less obvious, that NC holds even if some eigenvalues of A are purely imaginary (unitary magnitude in discrete time) ; the difficulty in principle is in the control of the possible polynomially unstable modes, using bounded inputs.

This more general statement has been proved for continuous time scalar $(m = 1)$ systems in Lee and Markus (1968), which also gives a proof in the (continuous) case with $m > 1$ but with extra assumptions on A. These assumptions were dropped in later works (Brammer 1972, Schmitendorf and Barnish 1980). One goal of this paper is to show that this result can be obtained as a simple consequence of some general facts (to be developed later) in the module theoretic theory of linear systems together with a rather interesting lemma about polynomials. More importantly, the proof given here applies equally well to (and in fact depends on) the discrete time case. The latter is of interest in itself and also in terms of the continuous time case, since it provides a result on sampled controllability. We present also the corresponding results for asymptotic null-controllability (ANC).

Finally, we shall mention an application to the problem of (local) non-linear ANC. It is desirable (Sontag 1983 a, b) to include in the definition of ANC, a statement about magnitudes of controls needed to control small states. The linear results mentioned above can be translated into one such statement.

Received 24 May 1983.

† Research supported in part by US Air Force Grant AFOSR-80-0196 and presented at the I.E.E.E. Conference on Decision and Control, December 1982.

‡ Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903, U.S.A.

2. Main results

Let $A \in R^{n \times n}$, $B \in R^{n \times m}$. Associate to (A, B) the continuous (resp., discrete) time system Σ with equations

$$\dot{x}(t)[x(t+1)] = Ax(t) + Bu(t) \tag{1}$$

Let $\phi(t, x, u)$ denote the state of Σ at time $t \geqslant 0$ which results from $x(0) = x$ and the application of the control $u(\cdot)$ (discrete or continuous time depending on the context; in the continuous case, take the controls to be piecewise continuous). Fix $U \subseteq R^n$. An (m-input) system Σ is in ANC (U) (asymptotically null-controllable using controls in U) if and only if, for each state x, there is an input $u(\cdot)$ with $u(t) \in U$ for all t and such that $\phi(t, x, u) \to 0$ as $t \to \infty$. If, moreover, there is for each x a u with values in U such that $\phi(T, x, u) = 0$ for some (finite) T, then Σ is in NC (U). We say that Σ is null-controllable (NC) (resp., ANC) if $\Sigma \in$ NC (R^n) (resp., $\Sigma \in$ ANC(R^n)), and that Σ is SINC (small-input NC) (resp. SIANC—small-input ANC) if and only if Σ is in NC(U) (resp., ANC(U)) for every U that is a neighbourhood of 0 in R^m. Finally, a continuous (resp., discrete) time Σ (or the corresponding A) is *asystable* if and only if each eigenvalue λ of A has Re $\lambda < 0$ (resp., $|\lambda| < 1$) and *adequate* if and only if each λ satisfies Re $\lambda \leqslant 0$ ($|\lambda| \leqslant 1$).

Factor the characteristic polynomial of A as $\pi = \pi_s \pi_u$, where π_u collects all the roots with Re $\lambda \geqslant 0$ ($|\lambda| \geqslant 1$, in discrete time). Let $X_u = X_u(A) := \ker \pi_u$, $A_u : X_u \to X_u$ be the restriction of A, and $B_u : R^m \to X_u$ the (co-)restricted map. There results a system Σ_u (well-defined up to a choice of basis for X_u). Similarly for a system Σ_s. Recall that Σ is ANC if and only if Σ_u is reachable. This is proved by a standard argument, as follows. If Σ is ANC consider the induced system in X_u/R_u (R_u being the reachable set of Σ_u). It is easy to see that this system is again ANC; on the other hand, this quotient system has $B = 0$ and an A which is not asystable, a contradiction. For the converse, note that reachability implies that Σ_u is NC and that, in general, $\Sigma \in$ ANC(U) whenever $\Sigma_u \in$ NC(U), provided that $0 \in U$. This is because one may always apply first a control with values in U sending the X_u-coordinate of a given state to 0 (in finite time), and then concatenate this control with one (of infinite length) constantly zero.

Regarding NC, recall that this is equivalent in continuous time, to reachability, and in discrete time to the image of A^n being included in the reachable set (so that it is again equivalent to reachability if A is known to be nonsingular, for example for Σ_u).

The main results on SINC and SIANC are as follows.

Theorem 1

The following statements are equivalent:

(*a*) Σ is SINC;
(*b*) $\Sigma \in$ NC(U) for some bounded neighbourhood of 0 in U;
(*c*) $\Sigma \in$ NC(U) for some bounded U; and
(*d*) Σ is NC and adequate.

Theorem 2

The following statements are equivalent :

(*a*) Σ is SIANC ;
(*b*) $\Sigma \in \text{ANC}(U)$ for some bounded neighbourhood of 0 in U ;
(*c*) $\Sigma \in \text{ANC}(U)$ for some bounded U ; and
(*d*) Σ is ANC and adequate.

Note that (*d*) in Theorem 2, is equivalent to asking that Σ_u be reachable and adequate (i.e., Σ_u is SINC) so that in particular all eigenvalues λ of A_u have zero real part ($|\lambda| = 1$ for discrete time).

The proofs of the theorems will be given below but first consider the following feedback characterization, which is fairly obvious (using the control canonical form) in the scalar ($m=1$) case. Choose any norm on matrices $K \in R^{m\times n}$.

Theorem 3

Σ is SIANC if and only if the following property holds ; for each $\beta > 0$ there is a K such that $\|K\| < \beta$ and $A+BK$ is asystable.

Proof

Assume Theorems 1 and 2 hold and that the stated property holds. By continuity (on K) of the eigenvalues of $A+BK$, Σ (equivalently, Σ_u) must be adequate. And the property clearly implies ANC of Σ, so Theorem 2 applies. Conversely, assume that Σ is SIANC. Observe that if the result is proved for Σ_u then it will also be true for Σ ; this is because, if $A_u + B_u K_u$ is asystable, then $A+BK$ also is, for $K := (0, K_u)$. We use this observation to conclude that we may assume that Σ is reachable (since Σ_u is). Consider now the smooth map

$$\Lambda : G \to R^{n\times n} : (T, K) \to T(A+BK)T^{-1} \tag{2}$$

where $G := GL(R^n) \times R^{m\times n}$, seen as a product manifold. We calculate the differential of Λ at $e := (I, 0)$, where I is the identity : consider for any $(S, L) \in T_e G$ the curve $(e^{tS}, I+tL)$; an application of the Baker–Campbell–Hausdorff formula gives that

$$d_e\Lambda(S, L) = [S, A] + BL \tag{3}$$

where $[\cdot]$ denotes the Lie bracket on $R^{n\times n}$. It can be proved that every matrix can be written as $[S, L] + BL$, for fixed (A, B) and varying (S, L), if and only if (A, B) is reachable (see Hermann and Martin 1977). Thus, Λ is a submersion at e. By the implicit function theorem, Λ is open on a neighbourhood N of e. Pick $\beta > 0$, and restrict N so that $(T, K) \in N$ implies $\|\tilde{K}\| < \beta$. Since $\Lambda(N)$ is an open neighbourhood of A, it must contain some asystable matrix C. (This last statement can be proved in a variety of ways ; for instance, by a perturbation of the non-fixed coefficients of the rational canonical form of A. It is essential here, of course, that A is adequate.) Thus $C = T(A+BK)T^{-1}$, for suitable T and K such that $\|K\| < \beta$. It follows that $A+BK$ is also asystable as desired. □

It is interesting to remark that an argument analogous to the above can be used to give a proof of the pole-shifting theorem that does not require Heymann's lemma for the reduction to the scalar case : given a reachable (A, B) the same argument shows that we may perturb A by feedback into a cyclic matrix ; now we know that there is a u such that (A, Bu) is reachable (see Wonham 1974, p. 42), and we are back in the scalar case.

We now turn to the proofs of Theorems 1 and 2. Note that $(a) \Rightarrow (b) \Rightarrow (c)$ are trivial in both. The implication $(d) \Rightarrow (a)$ in Theorem 2 follows from Theorem 1 : choose U a neighbourhood of zero. Since Σ_u is SINC, $\Sigma_u \in \mathrm{NC}(U)$, so by a previous remark we conclude that $\Sigma \in \mathrm{ANC}(U)$, as desired. Consider now $(d) \Rightarrow (a)$ in Theorem 1. We claim that it is enough for this to prove the result for reachable Σ. Indeed, assume given any adequate system Σ which is also NC. In continuous time, Σ is necessarily reachable. In discrete time, split A into a direct sum of a nilpotent A_0 and a non-singular A_1. The induced (A_1, B_1) is reachable, and hence also SINC if the result has been proved for reachable systems. But then Σ itself is SINC, because any controlling input for the non-singular subsystem can be concatenated by a finite sequence of zeros which controls the nilpotent part to zero. Further, we claim that for $(d) \Rightarrow (a)$ it is enough to treat the discrete time case (which is the topic of the next section). For this, assume that Σ is continuous time, reachable and adequate. Find a sampling rate d small enough that the induced discrete time system Σ_d is itself reachable. Since $A_d = \exp(dA)$ is (discrete time) adequate, Σ_d satisfies (d) and is therefore SINC. Thus Σ is itself SINC, in fact with sampled (i.e. constant on intervals of length d) controls.

Finally, we prove that $(c) \Rightarrow (d)$ in Theorems 1 and 2. We shall only give details for the continuous time case ; the discrete case is totally analogous. Since $\mathrm{NC}(U) \subseteq \mathrm{ANC}(U)$ for all U, all we need to prove is that (c) in Theorem 2 implies that Σ is adequate. We proceed analogously to Lee and Markus (1968, p. 92). Assume that Σ is not adequate. Modulo $GL(n)$, we may write the equations for Σ in such a way that the first coordinate is

$$\dot{x}_1 = \lambda x_1 + \Sigma b_i u_i \tag{4}$$

with some λ (real) > 0, or the first two coordinates are

$$\left.\begin{aligned} \dot{x}_1 &= \alpha x_1 + \beta x_2 + \Sigma b_i u_i \\ \dot{x}_2 &= -\beta x_1 + \alpha x_2 + \Sigma b_i u_i \end{aligned}\right\} \tag{5}$$

with real $\alpha > 0$. Since the U in (c) in Theorem 2 is bounded, there is a large enough initial $x_1(0)$ in (4) or a pair $(x_1(0), x_2(0))$ with $x_1{}^2(0) + x_2{}^2(0)$ large enough in (5), such that the derivative $\dot{x}_1$ (or the derivative of $x_1{}^2 + x_2{}^2$) is positive for any control with values in U. Thus there are initial vectors $x(0)$ which cannot be controlled asymptotically to zero, a contradiction.

3. Proof of sufficiency

We establish in this section the only remaining (and the most interesting) implication of Theorems 1 and 2 : if Σ is discrete time, reachable, and adequate then it is SINC. The proof will proceed by first reducing to the scalar case, through the introduction of a set of 'finite memory' input transformations which acts on reachable systems. These transformations will be such that systems in the same orbit will be simultaneously SINC and such that every system becomes equivalent to a parallel connection of scalar systems. (Note that the latter requirement would be satisfied by the feedback group, but the first requirement would not, since the SINC property is not preserved under feedback.)

Specifying a system (A, B) (with m inputs) amounts (up to isomorphism) to giving a pair (X, g), where X is a finitely generated torsion $R[z]$-module and $g : \Lambda \to X$ is a surjective $R[z]$-linear map (here we denote $\Lambda := R[z]^m$) (see, for instance, Kalman *et al.* 1969, chap. 10). We define an equivalence among reachable systems using their representation as pairs $(X, g) : \Sigma_1 \sim \Sigma_2$ will mean that there exist $R[z]$-linear maps $V : \Lambda \to \Lambda$ and $T : X \to X$ such that the following diagram commutes

$$\begin{array}{ccc} \Lambda & \xrightarrow{V} & \Lambda \\ g_1 \downarrow & & \downarrow g_2 \\ X_1 & \xrightarrow[T]{} & X_2 \end{array} \tag{6}$$

Another way of expressing this equivalence is the following. For a given automorphism V of Λ, there is a T as here if and only if $V(\ker g_1) = \ker g_2$. Thus two systems are equivalent if and only if the kernels of their g-maps are isomorphic as submodules of Λ; if D_i is a (square) matrix whose columns form a basis of $\ker g_i$, this amounts to requiring that D_1 and D_2 be equivalent as polynomial matrices.

Note the following facts: (*a*) Σ_1 isomorphic to Σ_2 implies $\Sigma_1 \sim \Sigma_2$ (isomorphism is the same as the existence of a T so that (identity, T) is as above); (*b*) when $m=1$, $\Sigma_1 \sim \Sigma_2$ only if they are isomorphic (because in that case V must be a multiplication by a non-zero scalar r, so that $r^{-1}T$ gives an isomorphism); (*c*) $\Sigma_1 \sim \Sigma_2$ implies that A_1 is similar to A_2, and in particular that Σ_1 is adequate if and only if Σ_2 is; and (*d*) for each Σ_1 there is a Σ_2 equivalent to it which is a parallel connection of scalar systems, i.e. such that the basis matrix D_2 of $\ker g_2$ is diagonal (this by existence of Smith forms, i.e. the fundamental structure theorem for finitely generated modules over a p.i.d.). Finally, we need the following property.

Lemma

Assume that $\Sigma_1 \sim \Sigma_2$. Then Σ_1 is SINC if and only if Σ_2 is SINC.

Proof

Assume that Σ_1 is SINC. Let U_2 be a neighbourhood of 0 in R^m, and pick an $\alpha > 0$ such that $\{\|u\| < \alpha\} \subseteq U_2$. We will prove that $\Sigma_2 \in \mathrm{NC}(U_2)$. Let (V, T) give the equivalence $\Sigma_1 \sim \Sigma_2$, and write $V = \Sigma V_i z^i$ as a polynomial of degree r with coefficients in $R^{m \times m}$. Pick any $\beta > 0$ such that $r\|V_i\|\beta < \alpha$ for $i = 1, \ldots, r$. (Use any pair of compatible norms on vectors and operators.) Take $U_1 := \{\|u\| < \beta\}$. Pick any $y \in X_2$ and let $x := T^{-1}y$, $x' := z^r x$. Since $\Sigma_1 \in \mathrm{NC}(U_1)$, there is an $\alpha > 0$ and a $w = \Sigma u_j z^j \in \Lambda$ of degree less than s such that

$$z^s x' + g_1(w) = 0 \tag{7}$$

Applying T to both sides of this equation gives

$$z^{r+s}y+g_2(w')=0 \tag{8}$$

where $w'=Vw$. Since $\deg w'<r+s$, (8) shows that w' controls y to 0. Write $w'=\Sigma u'_k z^k$. From the choice of β it follows that all $\|u'_k\|<\alpha$, so that w' has values in U_2, as desired. □

It follows from the above observations that it is enough to prove the result for systems which are parallel combinations of scalar systems, since all desired properties are preserved under equivalence. But then, it is only necessary to treat the scalar case, since a sum of systems is SIANC if and only if each system is (just apply independent controls on each channel). Consider the following fact about polynomials.

Proposition 1

Let $p(z)$ be a monic real polynomial all whose roots have magnitude less than or equal to one. Then, for each $\beta>0$, there exists a monic real polynomial $q(z)=z^k+\Sigma a_i z^i$ such that (1) p divides q, and (2) $|a_i|<\beta$ for each i.

Assume for a moment that the above result is true. Let $\Sigma=(X,g)$ be a (reachable) scalar adequate system. We shall identify X with $\Lambda/(p)$, where p is the characteristic polynomial of A, and g with the canonical quotient map. Pick now an $U\subseteq R$, containing a neighbourhood $\{|u|<\alpha\}$. Pick any state x. Let v_i, $i=0,\ldots,n-1$, be such that

$$z^{n-1}x=\Sigma v_i z^i (\mathrm{mod}\ p) \tag{9}$$

Let M be an upper bound on the $|v_i|$, and let $\beta:=\alpha(Mn)^{-1}$. Apply the above lemma to these p, β, and let q be as there. Thus, $z^k=-\Sigma a_i z^i (\mathrm{mod}\ p)$. It follows that

$$z^{n+k-1}=-(\Sigma a_i z^i)(\Sigma v_j z^j)=\Sigma c_h z^h (\mathrm{mod}\ p) \tag{10}$$

where the latter polynomial has degree $\leqslant n+k-2$. Thus x is controllable to zero using the inputs c_h. By the choice of β, all the c_h are in U, as desired. □

Proof of Proposition 1

Without loss of generality, we may assume that $\beta<1$. Note first that it is sufficient to find a polynomial q with *complex* coefficients satisfying the conclusions. This is because if p divides such a q, then it also divides q_1, where $q(z)=q_1(z)+iq_2(z)$ with both q_i real. We shall establish the result by induction on r (degree of p). Introduce for integers $k>0$ and complex numbers λ with $|\lambda|<1$, the polynomials

$$p_{k,\lambda}(z):=z^k-\Sigma(\lambda^{k-i}/k)z^i. \tag{11}$$

Note that λ is a root of $p_{k,\lambda}$, and that all its coefficients have magnitude $\leqslant 1/k$. If $p(z)=z-\lambda$ has $r=1$, pick any k with $1/k<\beta$; then $q:=p_{k,\lambda}$ is as desired. Assume now that $p(z)=(z-\lambda)p'(z)$, and that there is a monic polynomial q' with non-leading coefficients of magnitude $<\beta$ and divisible by p'. Pick now an integer $k>\deg q'$ such that $1/k<\beta$. Let $a:=\lambda^{k+1}$, and take

$$q''(z):=p_{a,k}(z^{k+1}) \tag{12}$$

Note that λ is a root of q'', so $z-\lambda$ divides q''. It follows that p divides $q := q'q''$. It only remains to see that all non-leading coefficients of q have magnitudes less than β. But, by construction, these coefficients are products (not sums of products) of the non-zero coefficients of q' and q'', for which the magnitudes are less than β. □

4. A (local) non-linear property

We wish to interpret here some of the previous concepts and results in terms of local properties of non-linear systems. Consider a continuous (resp., discrete) time system on R^n of the form

$$\dot{x}(t)[x(t+1)] = f(x(t), u(t)) \tag{13}$$

where f is differentiable with $f(0, 0) = 0$: $f(x, u) = Ax + Bu + o(x, u)$. Again assume controls are piecewise continuous in the continuous time case, and denote solutions by $\phi(t, x, u)$. Let Σ be the continuous (resp., discrete) time linear system associated with (A, B).

Proposition 2

Assume that Σ is SIANC. There exists then a neighbourhood V of 0 in R^n and a map $g : R_+ \to R_+$, such that (1) $g(\alpha) = o(\alpha)$ as $\alpha \to 0$, and (2) for each $x \in V$ there is a control u such that $\|u(t)\| < g(\|x\|)$ for all t and such that $\phi(t, x, u) \to 0$ as $t \to \infty$.

Proof

Through sampling at a suitable frequency, it is again sufficient to prove the result for discrete time systems. We claim first that, for each $\beta > 0$, there are an integer N, and N $m \times n$-matrices K_i, $i = 0, \ldots, N-1$, such that (a) all $\|K_i\| \leqslant \beta$; and (b) $\|A^N + \Sigma A^j BK_j\| < 1/4$. (It will be convenient here to use the 'L_1' norm $\|x\| = \Sigma |x_i|$ on R^n and R^m, and the corresponding operator norms.) To prove this claim, consider the canonical basis vectors e_i, and find an N, and for each i, a sequence of control values $u_0, \ldots, u_{N-1}$ such that $\|A^N e_i + \Sigma A^j Bu_j\| < 1/4$ and all the $\|u_j\| < \beta$. Then the matrix K_j with columns $u_j^{(1)}, \ldots, u_j^{(n)}$ has the desired properties. Consider now the (discrete) system (13), and take any state x. Pick $\beta > 0$ and obtain the K_j as above. Apply the control $u(N-j) := K_{j-1}x$, $j = 1, \ldots, N$. Then

$$\phi(N, x, u) = (A^N + \Sigma A^j BK_j)x + o(x) \tag{14}$$

We can then define functions $a : R_+ \to R_+$ and $N : R_+ \to \mathbf{N}$, such that the following property holds for each $\beta > 0$: For each x with $\|x\| < a(\beta)$ there is a control $u(\cdot)$ such that $\|u(t)\| < \beta\|x\|$ for each t and $\|\phi(N(\beta), x, u)\| < \|x\|/2$. Define by induction $\alpha_0 := a(1)$, and $\alpha_{k+1} := a(\alpha_k/k)$. Modify if necessary the α_i so that they monotonically decrease to 0 and $\alpha_0 < 1$. Let V be the ball of radius α_0 centred at 0. Assume that a state x is such that $\|x\| < \alpha_k$ for some k. Then there are an N and a control $u(\cdot)$ such that $\|u(j)\| < \alpha_{k-1}\|x\|/k < \|x\|$ for all j and such that $x' := \phi(N, x, u)$ has norm $< \alpha_k/2$. Since in particular $\|x'\| < \alpha_k$, we may repeat the construction and control x' to a state x'' with norm $< \alpha_k/4$, using another control with values of magnitude less than $\|x\|/k$. Iterating, we obtain by concatenation an infinite length control $u(\cdot)$ with each $\|u(t)\| < \|x\|/k$ and such that $\phi(t, x, u) \to 0$ as $t \to \infty$. For a map satisfying the requirements of the proposition, take now the piecewise linear map g which is linear in between the α_i and which has values $g(\alpha_k) := \alpha_k/(k-1)$. □

The converse of Proposition 2 is ' almost ' true. Specifically, the existence of V, g as there implies that the linearization Σ is adequate. In general, of course, it does not follow that Σ is ANC, but if the system (13) is itself linear (hence the same as Σ), then the conclusions imply local ANC hence by linearity global ANC, and so the converse holds. The proof that Σ must be adequate is analogous to that of the implication $(c) \Rightarrow (d)$ in Theorems 1 and 2. We sketch the case of continuous time systems with at least one positive real λ ; the other cases are similar. Choose coordinates so that the first equation of (13) is

$$\dot{x}_1 = \lambda x_1 + \Sigma b_i u_i + o(x, u) \tag{15}$$

Pick $\alpha > 0$ small enough so that $\|x\| < \alpha$ and $\|u\| < \alpha$ imply all of the following : $|o(x, u)| < \lambda\|x\|/2 + \|u\|$, $x \in V$, $g(\|x\|) < \|x\|$, and $(1 + \Sigma|b_i|)g(\|x\|)/\|x\| < \lambda/2$. Pick now any non-zero vector of the type $a := (a_1, 0, \ldots, 0)'$ with $\|a\| = a_1 < \alpha$. Find $u(\cdot)$ as in Proposition 2 ; thus $\|u(t)\| < g(a_1) < \alpha$ for all t and $x(t) := \phi(t, a, u)$ converges to zero. Without loss of generality, we may assume that $u(t)$ is continuous on t. Pick any $t \geqslant 0$ for which $x_1(t) = a_1$. The first equation gives

$$\dot{x}_1(t) = a_1(\lambda + c),$$

where

$$c := [\Sigma b_i u_i(t) + o(a, u(t))]/a_1.$$

But

$$|c| \leqslant \Sigma|b_i|g(a_1)/a_1 + \lambda\|a\|/2a_1 + g(a_1)/a_1 < \lambda \tag{16}$$

Thus $\dot{x}_1(t) > 0$ whenever $x_1(t) = a$. It follows that $x_1(\cdot)$ cannot satisfy $x(0) = a$ and also converge to zero, a contradiction. □

References

Brammer, R., 1972, *SIAM Jl. Control*, **10**, 339.

Hermann, R., and Martin, C., 1977, *Algebro-geometric and Lie Techniques in Systems Theory* (Brookline : Math. Sci. Press).

Kalman, R. E., Falb, P. L., and Arbib, M. A., 1969, *Topics in Mathematical System Theory* (New York : McGraw-Hill).

Lee, E. B., and Markus, L., 1968, *Foundations of Optimal Control Theory* (New York : Wiley).

Schmitendorf, W. E., and Barmish, B. R., 1980, *SIAM Jl. Control Optim.*, **18**, 327.

Sontag, E. D., 1983 a, *Inf. Control*, **51**, 105 ; 1983 b, *SIAM Jl Control Optim.*, **21**, 462.

Wonham, W. M., 1974, *Linear Multivariable Control* (New York : Springer).