Input Classes for Identifiability of Bilinear Systems

Eduardo D. Sontag, *Fellow, IEEE*, Yuan Wang, *Senior Member, IEEE*, and
Alexandre Megretski, *Senior Member, IEEE*

Abstract—This paper asks what classes of input signals are sufficient in order to completely identify the input/output behavior of generic bilinear systems. The main results are that step inputs are not sufficient, nor are single pulses, but the family of all pulses (of a fixed amplitude but varying widths) do suffice for identification.

Index Terms—Bilinear systems, identifiability, input/output equivalence.

Fig. 1. Identifying from i/o data.

I. INTRODUCTION

IN this paper, we address the following question: *what types of input signals are sufficient to completely identify the i/o behavior of a system?* In other words, we look for classes $\mathcal{U}$ of inputs with the property that, if a system σ is stimulated with the inputs from the set $\mathcal{U}$ and the corresponding time record of outputs is recorded, then, on the basis of the collected information on inputs and outputs it is possible—at least theoretically, with no regard to computational effort, and in the absence of noise—to obtain a system $\widehat{\sigma}$ which is equivalent to σ (Fig. 1). By "equivalent" we mean that the estimated system $\widehat{\sigma}$ will be completely indistinguishable from the true system σ in its i/o behavior, even when presented with inputs that do not belong to the restricted class $\mathcal{U}$ used for the identification experiments. Whether a certain class $\mathcal{U}$ of inputs is rich enough for identification is heavily dependent, of course, on prior assumptions about the system σ.

It is often very difficult to perform experiments in which arbitrary input profiles are used. Often, the only possible experiments are those in which steps, i.e. constant inputs, are applied. For example, in molecular biology, a step input corresponds to subjecting a cell culture to a fixed concentration of an extracelular ligand such as a drug or growth factor. Sometimes somewhat more complex inputs, such as pulses (keep the input constant at some level, then change it back to some default value) can be used, but this is already not easy to implement, much less more complicated test signals. This presents a theoretical challenge: how does one know if all possible "identifiable" information about the system can be obtained from such
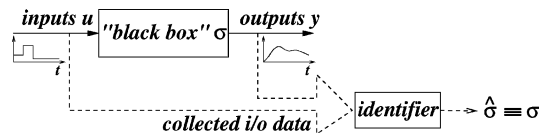
a restricted class of experiments? In the case of *linear* models of systems, this issue does not arise, because basically any single input, as long as it is nonzero, for example a single step or a single pulse, suffices for identification (or several inputs, one for each input channel, if the system has multiple inputs; for simplicity we restrict here to single-input systems). Note that we are thinking here of an experimental setup in which observations are collected over time. If, instead, only steady-state behavior was observed, and not transients, then one input is not enough, even for single-input linear systems. In that case, one has to use multiple inputs, such as steady-state measurements of responses to periodic signals at different frequencies. There has been work in the linear systems identification literature regarding conditions on inputs (persistence of excitation) for transient problems, in the context of adaptive control [1]. In the nonlinear systems literature, methods from differential algebra can be employed in order to characterize classes of inputs that avoid singularities leading to non-identifiability [2]. The present paper poses some basic theoretical questions related to the problem.

For nonlinear systems, it is thus an interesting question whether constant inputs or pulse inputs, or simple combinations of these input classes, suffice for identification, as they do for linear systems. In this paper, we show that for a large and interesting class of nonlinear systems, that of *bilinear systems*, constant inputs do not suffice, but pulses do.

Bilinear systems constitute an appealing class of nonlinear systems [3]–[6]. While for linear systems the evolution of the states is only allowed to depend on linear functions of the state variables and inputs, in bilinear systems one also allows a linear dependence on products between input and state variables. Bilinear systems can be easily described in linear-algebraic terms, and a theory, in many respects analogous to the linear theory, can be developed for the analysis of their input/output properties. On the other hand, bilinear systems are theoretically capable of approximating arbitrary input/output behaviors on finite time-intervals [7]–[9]. They have been used to model chemical processes, electrical networks, power plants, nuclear reactors, robotic manipulators, and many other systems in engineering, chemistry, biology, economics, and other fields [4]. They can also be employed in order to model and analyze certain simple enzymatic signaling cascades, when substrates are not too close to saturation and thus Michaelis-Menten kinetics can be replaced by bilinear expressions [10], [11].

E. D. Sontag is with the Department of Mathematics, Rutgers University, New Brunswick, NJ 08903 USA (e-mail: sontag@math.rutgers.edu).

Y. Wang is with the Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, FL 33431 USA (e-mail: ywang@math.fau.edu).

A. Megretski is with the Department of Electronic Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: ameg@mit.edu).

Informally (see next section for precise definitions and statements), the main results that we prove are as follows. On the negative side, we show that step inputs are not enough for identifying bilinear systems, nor do single pulses suffice. On the positive side, we show that the family of all pulses (of a fixed amplitude but varying widths) do suffice for identification. To be precise, one must impose certain non-degeneracy conditions on the classes of systems being considered, and, for the negative result, one wants to avoid trivial counter-examples in order to say something interesting. Thus, all results are stated for generic classes of systems.

Our techniques are based on realization theory. We make heavy use of ideas originally developed by Kalman for realizations by linear systems, and refined by Isidori and later Fliess for bilinear systems.

The organization of this paper is as follows. Section II gives the basic definitions, and reviews the background from bilinear realization theory. We provide a self-contained discussion because, even though the results proved in that section are not new, it is hard to find references presented as needed here. The main results are stated in Section III. The proofs of the negative results are given in Section IV and the proofs of the positive results are given in Section V. The latter are inspired by Juang's very nice paper [12] (we thank David Angeli for bringing this paper to our attention). Although the bilinear identification algorithm given in [12] involves some ambiguous steps, such as taking (non-unique) logarithms of matrices, we were able to adapt many of its basic ideas; we discuss in Section VI connections to that work. Conclusions and some remarks are presented in Section VII.

II. PRELIMINARIES

Systems and I/O Equivalence: Our results will be for bilinear systems, which are defined by affine vector fields on $\mathbb{R}^n$ and hence are described by matrices, but an abstract setup will allow us to discuss some preliminary facts in more generality. We consider single-input single-output initialized systems σ, in the usual sense of control theory [13]:

$$\dot{x}(t) = f_0(x(t)) + u(t)f_1(x(t)), \quad x(0) = x_0$$
$$y(t) = h(x(t)) \tag{1}$$

(we will drop the arguments "(t)" if clear from the context), where f_0 and f_1 are smooth vector fields on a manifold M and h is an output function $M \to \mathbb{R}$. Inputs can be taken to be any Lebesgue-measurable, essentially bounded, functions $u : [0, T_u] \to \mathbb{R}$, but there will be no difference in results if one restricts to, for instance, piecewise continuous inputs. We let Ω be the set of all inputs. In principle, solutions are unique but only defined on some maximal interval (which depends on the initial condition and the input), but for simplicity, and since it is the case anyway for bilinear systems, we assume that solutions are defined for all times $t \in [0, T_u]$ (forward completeness). We let $\varphi(t, u)$, or just $x(t)$ if the input is clear from the context, be the solution of (1) at time t, and $y(t) = h(\varphi(t, u))$ the corresponding output. When more than one system is being considered, we use appropriate notations; for example, a system $\widehat{\sigma}$

will be described by equations $\dot{x} = \widehat{f}_0(x) + u\widehat{f}_1(x)$, $x(0) = \widehat{x_0}$, $y = \widehat{h}(x)$.

Given two systems $\sigma, \widehat{\sigma}$, and an input u, we say that $\sigma, \widehat{\sigma}$ are *indistinguishable* under u if $h(\varphi(t, u)) = \widehat{h}(\widehat{\varphi}(t, u))$ for all $t \in [0, T_u]$. If $h(\varphi(t, u)) \neq \widehat{h}(\widehat{\varphi}(t, u))$ for some t, we say that u *distinguishes* among the two systems $\sigma, \widehat{\sigma}$; in other words, the "input/output experiment" consisting of perturbing the system with this input u results in a different time-varying observation for σ than for $\widehat{\sigma}$.

Given a subset $\mathcal{U} \subseteq \Omega$ of inputs, we say that the two systems $\sigma, \widehat{\sigma}$ are *input/output (i/o) equivalent with respect to all inputs in $\mathcal{U}$* if no input in $\mathcal{U}$ is able to distinguish between the two systems $\sigma, \widehat{\sigma}$, i.e., provided that $h(\varphi(t, u)) = \widehat{h}(\widehat{\varphi}(t, u))$ for each $u \in \mathcal{U}$ and for each $t \in [0, T_u]$. We write $\sigma \underset{\mathcal{U}}{\equiv} \widehat{\sigma}$ in that case.

In the special case in which $\mathcal{U} = \Omega$, we write simply $\sigma \equiv \widehat{\sigma}$ and simply say that the two systems are i/o equivalent. That is to say, they cannot be distinguished in any way whatsoever based on their "back box" input/output behavior.

Let Σ be a class of systems. A subset $\mathcal{U} \subseteq \Omega$ of inputs is said to be *sufficient for identifying systems in the class Σ* if, for any pair of systems $\sigma, \widehat{\sigma}$ in Σ,

$$\sigma \underset{\mathcal{U}}{\equiv} \widehat{\sigma} \Rightarrow \sigma \equiv \widehat{\sigma}.$$

In other words: whenever σ is not i/o equivalent to $\widehat{\sigma}$, there must exist some input in the set $\mathcal{U}$ which distinguishes among the two systems $\sigma, \widehat{\sigma}$.

Linear systems (finite-dimensional, continuous-time) are those for which f_0 is linear, f_1 is constant, and h is linear, i.e. systems described by equations

$$\dot{x} = Ax + bu, \quad x(0) = 0, \quad y = cx \tag{2}$$

with $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, and $c \in \mathbb{R}^{1 \times n}$. We often refer interchangeably to a linear system or its corresponding triple of matrices (A, b, c). Linear systems can be identified by any single nonzero input on a nontrivial interval, such as a constant function (a step), or a pulse.

4-Tuples and Bilinear Systems: We consider two slightly different classes of bilinear systems. To define these classes, we first introduce *4-tuples* as follows:

$$(A, N, b, c) \text{ where } A, N \in \mathbb{R}^{n \times n}, \ b \in \mathbb{R}^{n \times 1}, \ c \in \mathbb{R}^{1 \times n} \tag{3}$$

(the integer n is called the *dimension* of the 4-tuple).

We say that a system (1) is a *bilinear system of type I* if f_0 is linear, f_1 is affine, $x_0 = 0$, and h is linear. In other words, the system equations are:

$$\dot{x} = (A + uN)x + bu, \quad x(0) = 0$$
$$y = cx \tag{4}$$

where (A, N, b, c) is some 4-tuple as in (3). We use a notation such as "σ^o" to refer to a system of type I. With some abuse of terminology, we also simply write $\sigma^o = (A, N, b, c)$. Note that linear systems (2) constitute the subclass of bilinear systems of type I for which $N = 0$.

We say that a system (1) is a *bilinear system of type II* if f_0 and f_2 are both linear and h is linear (but the initial state may be nonzero). In other words, the system equations are:

$$\dot{x} = (A + uN)x, \quad x(0) = b$$
$$y = cx \tag{5}$$

where (A, N, b, c) is a 4-tuple as in (3). Once again, we do not differentiate between a system of type II and its associated 4-tuple when the meaning is clear from the context.

The key idea of the negative results is, taking the second class of systems as an example, as follows. Constant inputs $u(t) \equiv \beta$ allow one to determine the moments $c(A + \beta N)^k b$ for all integers k, and for all β. When $\beta = 0$, we can identify the triple (A, b, c) up to a change of state variables. However, this is not enough information for identifying N, which is a free parameter after a basis has been chosen in state space.

Similarity: We say that two 4-tuples (A, N, b, c) and $(\widehat{A}, \widehat{N}, \widehat{b}, \widehat{c})$ are *similar* if they have the same dimension n and there exists an invertible $n \times n$ matrix T such that the following equalities hold:

$$A = T\widehat{A}T^{-1}, \quad N = T\widehat{N}T^{-1}, \quad b = T\widehat{b}, \quad c = \widehat{c}T^{-1}. \tag{6}$$

Note that, for linear systems ($N = \widehat{N} = 0_{n \times n}$), this reduces to the familiar equivalence relation in linear systems theory.

We say that two bilinear systems $\sigma, \widehat{\sigma}$ (both of type I or both of type II) are *similar* (or "internally equivalent"), and we write

$$\sigma \sim \widehat{\sigma}$$

if there is a change of variables $x = Tz$ such that the equations of σ get transformed into those of $\widehat{\sigma}$. For systems of type I, this means that

$$(A + uN)x + bu = \dot{x} = T\dot{z} = T\left(\widehat{A} + u\widehat{N}\right)T^{-1}x + T\widehat{b}u$$

for all x and u, and also $cx = cTz = \widehat{c}z$; thus, $\sigma \sim \widehat{\sigma}$ is the same as saying that the 4-tuples (A, N, b, c) and $(\widehat{A}, \widehat{N}, \widehat{b}, \widehat{c})$ are similar. An analogous statement holds for systems of type II.

An easy calculation shows that $\sigma \sim \widehat{\sigma} \Rightarrow \sigma \equiv \widehat{\sigma}$, and a converse holds as well, under certain minimality assumptions, as discussed below.

Checking I/O Equivalence: For analytic systems, input/output equivalence can be verified by checking certain algebraic equalities, and there is no need to test all possible inputs, as we discuss next.

For any smooth vector field f on $\mathbb{R}^n$, and any smooth function $h : \mathbb{R}^n \to \mathbb{R}$, the Lie derivative $L_f h : \mathbb{R}^n \to \mathbb{R}$ is defined as the function $(L_f h)(x) = \nabla h(x) f(x)$, where ∇h is the gradient of h. (In differential-geometric terms, $L_f h$ is simply the value of the vector field on h, when vector fields are viewed as derivations on spaces of smooth functions.) More generally, if $f_1, \ldots, f_k$ are vector fields, the iterated derivative $L_{f_k} \ldots L_{f_1} h$ is defined recursively by the formula $L_{f_k}(L_{f_{k-1}} \ldots L_{f_1} h)$.

Suppose that σ and $\widehat{\sigma}$ are two systems (1) for which the vector fields f_0 and f_1 are analytic and the function h is also analytic. Then, $\sigma \equiv \widehat{\sigma}$ if and only if

$$\left(L_{f_{i_k}} \ldots L_{f_{i_1}} h\right)(x_0) = \left(L_{\widehat{f}_{i_k}} \ldots L_{\widehat{f}_{i_1}} \widehat{h}\right)(\widehat{x_0}) \tag{7}$$

for all sequences $(i_1, \ldots, i_k) \in \{0, 1\}^k$ and all $k \geq 0$. (When $k = 0$, (7) says that $h(x_0) = \widehat{h}(\widehat{x_0})$.) This is true because the expressions in (7) are the coefficients of the Fliess generating series of the input/output behavior associated to the respective systems, and the i/o behavior is in one-to-one correspondence with the coefficients of the series, see [14], Lemma 2.1.

For bilinear systems of types I or II, i/o equivalence amounts to an equality of vectors. Indeed, take first systems of type I. In this case, $f_0(x) = Ax$, $f_1(x) = Nx + b$, and $h(x) = cx$. Therefore, one can see inductively that:

$$\left(L_{f_{i_k}} \ldots L_{f_{i_1}} h\right)(x)$$
$$= \begin{cases} \text{if } i_k = 0: & cA_{i_1} \ldots A_{i_k} x \\ \text{if } i_k = 1: & cA_{i_1} \ldots A_{i_k} x + cA_{i_1} \ldots A_{i_{k-1}} b \end{cases}$$

where $A_0 = A$ and $A_1 = N$. In particular, for $x = x_0 = 0$, we have that $(L_{f_{i_k}} \ldots L_{f_{i_1}} h)(x) = 0$ for all sequences with $i_k = 0$, and $(L_{f_{i_k}} \ldots L_{f_{i_1}} h)(x) = cA_{i_1} \ldots A_{i_{k-1}} b$ for all sequences with $i_k = 1$.

Generally, given two 4-tuples (A, N, b, c) and $(\widehat{A}, \widehat{N}, \widehat{b}, \widehat{c})$, let us say that they are *i/o equivalent* if

$$cA_{i_1} \ldots A_{i_k} b = \widehat{c}\widehat{A}_{i_1} \ldots \widehat{A}_{i_k} \widehat{b} \tag{8}$$

for all sequences of matrices A_j picked out of A and N, including the "empty" sequence ($cb = \widehat{c}\widehat{b}$). (It suffices to check sequences of length $n + \widehat{n}$, where $n, \widehat{n}$ are the respective state-space dimensions; cf. [15]–[20].)

Then the preceding discussion proves:

Lemma 2.1: Two systems σ and $\widehat{\sigma}$ of type I are i/o equivalent if and only if the corresponding 4-tuples are i/o equivalent. □

For bilinear systems of type II, the same conclusion holds, in this case because

$$\left(L_{f_{i_k}} \ldots L_{f_{i_1}} h\right)(b) = cA_{i_1} \ldots A_{i_k} b.$$

Lemma 2.2: Two systems σ and $\widehat{\sigma}$ of type II are i/o equivalent if and only if the corresponding 4-tuples are i/o equivalent. □

Canonical Systems and Uniqueness: A 4-tuple (A, N, b, c) as in (3) will be said to be *canonical* provided that the following two properties hold:

1) There is no proper subspace of $\mathbb{R}^n$ that contains b and is invariant under $x \mapsto Ax$ and $x \mapsto Nx$.
2) There is no nonzero subspace of $\mathbb{R}^n$ that is contained in the nullspace of $x \mapsto cx$ and is invariant under $x \mapsto Ax$ and $x \mapsto Nx$.

The first property can be equivalently expressed by saying that the set of vectors of the form

$$A_{i_1} \ldots A_{i_k} b \tag{9}$$

ranging over all matrix products with $A_j \in \{A, N\}$ (including $k = 0$, i.e., b), or equivalently over all products of length k at most $n - 1$, must span all of $\mathbb{R}^n$. This property is often called "span-reachability" because, for bilinear systems, it amounts to the requirement that the set of states reachable from the origin span all of the state-space. Similarly, the second property can be equivalently expressed by the dual property that the span of the vectors

$$A'_{i_1} \ldots A'_{i_k} c'$$

(prime indicates transpose) be all of $\mathbb{R}^n$ (once again, length $\leq n - 1$ suffices), and is an observability property for bilinear systems. Canonical 4-tuples are also called "minimal" because [15]–[20] they have minimal dimension among all other 4-tuples which are i/o equivalent in the sense of (8); moreover, if a 4-tuple (A, N, b, c) is not canonical, then [15]–[20] there is some 4-tuple $(\widehat{A}, \widehat{N}, \widehat{b}, \widehat{c})$ which is canonical and is so that (8) holds. (We do not need in this paper the interpretations in terms of reachability and observability, nor the minimality result.) In the identification literature, "canonical" is often also referred to as "identifiable." We will call a bilinear system σ (of type I or II) canonical if the corresponding 4-tuple is canonical.

A very special case is that of linear systems (2), i.e. systems of type I with $N = 0$). Such a system $\sigma = (A, b, c)$ is canonical if and only if it is reachable and observable in the usual sense of control theory [13]. The controllability matrix $\mathcal{R}(A, b)$ and the observability matrix $\mathcal{O}(A, c)$ are defined respectively by:

$$\mathcal{R}(A, b) = \begin{pmatrix} b & Ab & \ldots & A^{n-1}b \end{pmatrix}$$
$$\mathcal{O}(A, c) = \mathcal{R}(A', c')' = \begin{pmatrix} c' & A'c' & \ldots & (A')^{n-1}c' \end{pmatrix}'$$

(prime indicates matrix transpose). The system σ is canonical iff both matrices have full rank n.

Similarity and I/O Equivalence: We already remarked that $\sigma \sim \widehat{\sigma} \Rightarrow \sigma \equiv \widehat{\sigma}$ for any two bilinear systems (both of the same type). Conversely, if both systems σ and $\widehat{\sigma}$ are canonical, $\sigma \equiv \widehat{\sigma} \Rightarrow \sigma \sim \widehat{\sigma}$. Thus:

$$\text{if } \sigma \text{ and } \widehat{\sigma} \text{ are canonical}, \quad \sigma \equiv \widehat{\sigma} \Longleftrightarrow \sigma \sim \widehat{\sigma}. \quad (10)$$

This is a standard fact about bilinear systems [15]–[20] (strictly speaking, these references deal with discrete-time systems such as $x(t + 1) = (A + u(t)N)x(t)$, but the algebraic statement about 4-tuples is the same as in the continuous-time case). The proof is, in fact, completely analogous to the proof for linear systems [13]. For completeness, we provide a proof here:

Lemma 2.3: Suppose that the two 4-tuples (A, N, b, c) and $(\widehat{A}, \widehat{N}, \widehat{b}, \widehat{c})$ are canonical and i/o equivalent. Then they are similar. Moreover, the similarity transformation T in (6) is unique.

Proof: Pick any $\widehat{x} \in \mathbb{R}^n$. By the span-reachability property (9) for $(\widehat{A}, \widehat{N}, \widehat{b}, \widehat{c})$, there are real numbers λ_α, where α denotes sequences $(i_1, \ldots, i_k)$ of length at most $n - 1$ (including the "empty" sequence) such that $\widehat{x} = \sum_\alpha \lambda_\alpha \widehat{A}_\alpha \widehat{b}$, where we denote $A_\alpha = A_{i_1} \ldots A_{i_k}$ for $\alpha = (i_1, \ldots, i_k)$. Now define $T\widehat{x} := \sum_\alpha \lambda_\alpha A_\alpha b$.

There are many possible representations of a vector $\widehat{x}$ as a linear combination of the spanning set in (9) for $(\widehat{A}, \widehat{N}, \widehat{b}, \widehat{c})$, so to see that T is well-defined as a mapping we need to verify that if $\sum_\alpha \lambda_\alpha \widehat{A}_\alpha \widehat{b} = \sum_\alpha \lambda'_\alpha \widehat{A}_\alpha \widehat{b}$ then $\sum_\alpha \lambda_\alpha A_\alpha b = \sum_\alpha \lambda'_\alpha A_\alpha b$.

By linearity, it is enough to show that $\sum_\alpha \lambda_\alpha \widehat{A}_\alpha \widehat{b} = 0 \Rightarrow \sum_\alpha \lambda_\alpha A_\alpha b = 0$. Suppose that $\sum_\alpha \lambda_\alpha \widehat{A}_\alpha \widehat{b} = 0$. Then also $\widehat{c} \widehat{A}_\beta \sum_\alpha \lambda_\alpha \widehat{A}_\alpha \widehat{b} = 0$ for any other index β, or equivalently $\sum_\alpha \lambda_\alpha \widehat{c} \widehat{A}_{\beta\alpha} \widehat{b} = 0$, where $\beta\alpha$ is the concatenation of the sequences β and α. Now, i/o equivalence of the two given 4-tuples implies that $\widehat{c} \widehat{A}_{\beta\alpha} \widehat{b} = c A_{\beta\alpha} b$ for all indices, and so also $cA_\beta x = \sum_\alpha \lambda_\alpha c A_{\beta\alpha} b = 0$. This holds for any index β, so, using the observability of (A, N, b, c), we conclude that $x = 0$, as desired. The mapping T is obviously linear (by definition), and it is onto because of the reachability of (A, N, b, c), which means that every $x \in \mathbb{R}^n$ can be written as $\sum_\alpha \lambda_\alpha A_\alpha b$ for some λ_α's. To prove that T is one-to-one, we simply reverse the argument used to prove that T was well-defined. Uniqueness follows by the same argument. ∎

By picking among all the possible linear combinations the one whose coefficients have minimal Euclidean norm, one obtains an explicit expression for T:

$$T = \mathcal{R} \widehat{\mathcal{R}}^{\#}$$

where # denotes matrix pseudoinversion, $\mathcal{R}$ is a matrix listing the products in (9) of length $\leq n - 1$, and $\widehat{\mathcal{R}}$ lists the vectors in the same order for the second 4-tuple. For linear systems $\sigma = (A, b, c)$, the equivalence becomes

$$T = \mathcal{R}(A, b) \mathcal{R}(\widehat{A}, \widehat{b})^{-1} \quad (11)$$

where R is the usual reachability matrix ([13], Theorem 27).

Generic Sets of Systems: We will make statements about "generic" classes of systems, so we must define this term carefully. Genericity can be defined in many ways, for example in probabilistic terms (a set is generic if it has "probability one") or, as usual in mathematics, in terms of open dense sets. In order to provide the strongest possible results, we combine both definitions and say here that a subset S of an Euclidean space $\mathbb{R}^\ell$ is *generic* provided that:

- the set S has full measure, that is, the complement S^c has Lebesgue measure zero, and
- the set S is open (and dense) in $\mathbb{R}^\ell$.

When dealing with sets of 4-tuples (3), we view such sets as subsets of $\mathbb{R}^\ell$ with $\ell = 2n^2 + 2n$.

When talking about genericity of classes of systems of type I or II, we mean genericity of the sets of associated 4-tuples. Specifically, if we let $\mathcal{S}_n^I$ be the class of n-dimensional bilinear systems of type I, then we think of $\mathcal{S}_n^I$ as $\mathbb{R}^{2n^2+2n}$, and similarly for the class $\mathcal{S}_n^{II}$ of n-dimensional bilinear systems of type II.

III. STATEMENTS OF MAIN RESULTS

For any $\alpha \in \mathbb{R}$ and any $\tau \geq 0$, let $\mathcal{I}_{\tau,\alpha}$ denote the class of all functions of the form

$$\mathbf{u}_{\tau,\alpha,\beta}(t) = \begin{cases} \alpha & \text{for } 0 \leq t < \tau \\ \beta & \text{for } t \geq \tau \end{cases}$$

where β is a constant. Note that in the special case when $\tau = 0$, $\mathcal{I}_{\tau,\alpha}$ becomes the class of constant functions. (There is a small ambiguity in that we have not specified the domain of the inputs. We can view these inputs as defined on some interval $[0, T]$ with $T > \tau$; any such T will give the same results.)

Let $\mathbf{u}_{\tau,\alpha}$ denote the particular pulse function in $\mathcal{I}_{\tau,\alpha}$ for which $\beta = 0$, that is,

$$\mathbf{u}_{\tau,\alpha}(t) = \begin{cases} \alpha & \text{for } 0 \le t < \tau \\ 0 & \text{for } t \ge \tau. \end{cases} \quad (12)$$

Let us now state the negative main results of this paper.

Theorem 1: For each $\tau \ge 0$ and $\alpha \in \mathbb{R}$, there is a generic subset $\mathcal{S}$ of $\mathcal{S}_n^I$ such that, for every system $\sigma^o \in \mathcal{S}$, there is some $\widehat{\sigma^o} \in \mathcal{S}$ such that
   1) σ^o and $\widehat{\sigma^o}$ are i/o equivalent under the pulse function $\mathbf{u}_{\tau,\alpha}(\sigma^o \underset{\{\mathbf{u}_{\tau,\alpha}\}}{\equiv} \widehat{\sigma^o})$, but
   2) σ^o and $\widehat{\sigma^o}$ are not i/o equivalent.

Theorem 2: For each $\tau \ge 0$ and $\alpha \in \mathbb{R}$, there is a generic subset $\mathcal{G}$ of $\mathcal{S}_n^{II}$ such that, for every system $\sigma \in \mathcal{G}$, there is some $\widehat{\sigma} \in \mathcal{G}$ such that
   1) σ and $\widehat{\sigma}$ are i/o equivalent under all the pulses in the set $\mathcal{I}_{\tau,\alpha}(\sigma \underset{\mathcal{I}_{\tau,\alpha}}{\equiv} \widehat{\sigma^o})$, but
   2) σ and $\widehat{\sigma}$ are not i/o equivalent.

By setting $\tau = 0$ for the collection $\mathcal{I}_{\tau,\alpha}$, one obtains the following as a consequence of Theorem 2:

Corollary 3.1: There is a generic subset $\mathcal{G}$ of $\mathcal{S}_n^{II}$ such that, for every system $\sigma \in \mathcal{G}$, there is some $\widehat{\sigma} \in \mathcal{G}$ such that
   1) σ and $\widehat{\sigma}$ are i/o equivalent under every constant input, but
   2) σ and $\widehat{\sigma}$ are not i/o equivalent. □

The first part of Corollary 3.1 may be restated as follows: for every $(A, N, b, c) \in \mathcal{G}$, there exists some $(\widehat{A}, \widehat{N}, \widehat{b}, \widehat{c}) \in \mathcal{G}$ such that

$$ce^{(A+\beta N)t}b = \widehat{c}e^{(\widehat{A}+\beta\widehat{N})t}\widehat{b}$$

for all $\beta \in \mathbb{R}$ and all $t \ge 0$. This implies that:

$$c\int_0^t e^{(A+\beta N)(t-s)}\beta b\, ds = \widehat{c}\int_0^t e^{(\widehat{A}+\beta\widehat{N})t}\beta\widehat{b}\, ds.$$

for all $\beta \in \mathbb{R}$ and all $t \ge 0$. Hence, the result in Corollary 3.1 also applies to systems as in (4):

Corollary 3.2: There is a generic subset $\mathcal{G}$ of $\mathcal{S}_n^I$ such that, for every $\sigma \in \mathcal{G}$, there is some $\widehat{\sigma} \in \mathcal{G}$ such that
   1) σ and $\widehat{\sigma}$ are i/o equivalent under every constant input, but
   2) σ and $\widehat{\sigma}$ are not i/o equivalent. □

Next, we state our positive results for systems of both types. For any $\alpha \in \mathbb{R}$, let $\mathcal{V}_\alpha$ denote the set of pulses of magnitude α:

$$\mathcal{V}_\alpha := \{\mathbf{u}_{\tau,\alpha} | \tau \ge 0\}.$$

Theorem 3: For each $\alpha \ne 0$, there is a generic subset $\mathcal{M}$ of $\mathcal{S}_n^I$ such that, for every pair of systems $\sigma^o, \widehat{\sigma} \in \mathcal{M}$,

$$\sigma^o \underset{\mathcal{V}_\alpha}{\equiv} \widehat{\sigma^o} \Longleftrightarrow \sigma^o \equiv \widehat{\sigma^o}.$$

Theorem 4: For each $\alpha \ne 0$, there is a generic subset $\mathcal{M}$ of $\mathcal{S}_n^{II}$ such that, for every pair of systems $\sigma^o, \widehat{\sigma} \in \mathcal{M}$,

$$\sigma^o \underset{\mathcal{V}_\alpha}{\equiv} \widehat{\sigma^o} \Longleftrightarrow \sigma^o \equiv \widehat{\sigma^o}.$$

A. Finite Numbers of Experiments

The positive results, Theorems 3 and 4, merely assert that inputs in $\mathcal{V}_\alpha$ are sufficient for identifiability, and no claim is made regarding how many such inputs are required, nor at how many time points does one need to observe the outputs. However, the results can be refined to show that $K = 2(n^2 + 2n) + 1$ data points suffice. More precisely, consider the set

$$\mathcal{P} = \left\{(\tau_1, t_1, \tau_2, t_2, \ldots, \tau_K, t_K) \in \mathbb{R}_+^{2K} \text{ such that } \tau_i \le t_i \forall i\right\}.$$

For each $\alpha \ne 0$, there is a generic subset $\mathcal{P}_\alpha$ of $\mathcal{P}$, and a generic class of systems $\mathcal{M}'$ of type I (or of type II), with the following property:

Suppose that $(\tau_1, t_1, \tau_2, t_2, \ldots, \tau_K, t_K) \in \mathcal{P}_\alpha$, and that σ^o and $\widehat{\sigma^o}$ are two systems in $\mathcal{M}'$ with the property that for all $i = 1, \ldots, K$ their outputs at time t_i are the same when the input is $\mathbf{u}_{\tau_i,\alpha}$. Then $\sigma^o \underset{\mathcal{V}_\alpha}{\equiv} \widehat{\sigma^o}$.

In particular, for systems in the generic class $\mathcal{M} \cap \mathcal{M}'$, it follows that $\sigma^o \equiv \widehat{\sigma^o}$.

To prove this fact, we let $r = n^2 + 2n$, and we view $p \in \mathbb{R}^r$ as parameterizing a quadruple $\phi(p) = (A, N, b, c)$, where the triple (A, b, c) is in control canonical form ($2n$ parameters) and N is arbitrary (n^2 parameters). We now define $\rho(p, \tau, t)$ as the output at time $t \ge \tau$ of the system defined by $\phi(p)$ and with input $\mathbf{u}_{\tau,\alpha}$. We view this mapping as defined on the set of pairs (τ, t) with $0 \le \tau \le t$. Note that this is a real-analytic function of (p, τ, t), since the formula defining the output of a system at time $t \ge \tau$ is real analytic; for example, for systems of type II one has

$$\rho(p, \tau, t) = ce^{(t-\tau)A}e^{\tau(A+\alpha N)}b$$

(for systems of type I, the expression is slightly more complicated), where $\phi(p) = (A, N, b, c)$. The domain of ρ is not open, but the mapping can be extended as an analytic mapping on $\mathbb{R}^r \times \mathbb{R}^2$ by using the same formula. Note that, by analyticity, $\rho(p_1, \tau, t) = \rho(p_2, \tau, t)$ for all $0 \le \tau \le t$ if and only if the same holds for the extended mapping, i.e. for all $(\tau, t) \in \mathbb{R}^2$.

By the main theorem for maps in [21], there is a generic subset $\mathcal{P}_\alpha$ of $\mathcal{P}$ (corresponding to a "universal distinguishing set" of experiments) such that, if $\rho(p, \tau_i, t_i) = \rho(\widehat{p}, \tau_i, t_i)$ for each $i = 1, \ldots, K$, then $\rho(p, \tau, t) = \rho(\widehat{p}, \tau, t)$ for every τ, t with $0 \le \tau \le t$. (Observe that the systems result in [21] requires analyticity, which does not hold for the outputs of systems with respect to switched controls such as pulses; hence the need to argue in terms of the mapping ρ.)

Now we let $\mathcal{M}'$ consist of systems of type I (or of type II) for which the triple (A, b, c) is canonical in the sense of linear systems. Let us consider two systems σ^o and $\widehat{\sigma^o}$ in $\mathcal{M}'$ that have the same outputs at the times t_i when the input is $\mathbf{u}_{\tau_i,\alpha}$, for every $i = 1, \ldots, K$.

For each system in $\mathcal{M}'$, there is a similarity bringing the triple (A, b, c) into control canonical form. So we can find two parameter vectors p and $\widehat{p}$ such that the respective systems are i/o equivalent to σ^o and $\widehat{\sigma^o}$. This means that $\rho(p, \tau_i, t_i) = \rho(\widehat{p}, \tau_i, t_i)$, and therefore, by the choice of $\mathcal{P}_\alpha$,

we have that $\rho(p, \tau, t) = \rho(\widehat{p}, \tau, t)$ for all τ and t. Now pick any $\tau \geq 0$ and consider the input $\mathbf{u}_{\tau,\alpha}$ and an arbitrary time $t \geq 0$. We claim that the systems σ^o and $\widehat{\sigma^o}$ have the same outputs at time t. There are two cases: $t < \tau$ and $t \geq \tau$. In the second case, the outputs coincide because they are respectively the same as $\rho(p, \tau, t)$ and $\rho(\widehat{p}, \tau, t)$, which are equal to each other. So, assume now that $t < \tau$. The key observation now is that the outputs at time t corresponding to σ^o and $\widehat{\sigma^o}$ when the input $\mathbf{u}_{\tau,\alpha}$ is used are exactly the same as the outputs at time t for these systems when the input $\mathbf{u}_{t,\alpha}$ is used instead. Since $t \leq t$, these outputs are the same as $\rho(p, t, t)$ and $\rho(\widehat{p}, t, t)$, which are equal. So $\sigma^o \underset{\mathcal{V}_\alpha}{\equiv} \widehat{\sigma^o}$, as claimed.

IV. PROOFS OF NEGATIVE RESULTS

A. Some Preliminaries

The construction in this section is key to the proofs of the negative results. The following observation was apparently first made in [22] (see problem 1 in page 110, and problem 12 in page 105). See the Appendix for a proof.

Lemma 4.1: For each canonical triple $\sigma = (A, b, c)$, there is a unique matrix $T = T(\sigma)$ such that

$$AT = TA', \quad b = Tc', \quad cT = b'. \tag{13}$$

Moreover, the matrix $T(\sigma)$ is given by $T(\sigma) = \mathcal{R}(A, b)$ $[(\mathcal{O}(A, c))']^{-1}$. □

For each nonzero $n \times n$ matrix S, consider the following set:

$$\mathcal{B}(S) := \{N \in \mathbb{R}^{n \times n} | NS = SN'\}.$$

Note that $\mathcal{B}(S)$ is a proper linear subspace of $\mathbb{R}^{n \times n}$, because, in particular, when E_{ij} is the matrix having a 1 in its (i, j)th position and zero elsewhere, $E_{ij}S = SE'_{ij}$ implies that $s_{kj} = 0$ for all $k \neq i$.

We now define a set $\mathcal{G}_0$ that will play a major role in the constructions. It is defined as the set consisting of those 4-tuples

$$(A, N, b, c) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$$

such that:

1) (A, b, c) is canonical,
2) $N \notin \mathcal{B}(T(A, b, c))$.

Note that, since the triple (A, b, c) is already canonical, every element of $\mathcal{G}_0$ is canonical as a 4-tuple.

For the next result, it is more elegant not to use inverses. Given a triple $\sigma = (A, b, c)$, let $\tilde{\mathcal{O}}$ denote the cofactor matrix of $\mathcal{O} = \mathcal{O}(A, c)$. Note that if σ is observable, then $\mathcal{O}^{-1} = (1/\Delta)\tilde{\mathcal{O}}$, where $\Delta(\sigma) = \det \mathcal{O}$.

For any triple $\sigma = (A, b, c)$ (not necessarily minimal), we define $\tilde{T}(\sigma) := \mathcal{R}(A, b)(\hat{\mathcal{O}}(A, c))'$. This is a polynomial expression on the entries of A, b, and c. If (A, c) is observable, $T(\sigma) = (1/\Delta(\sigma))\tilde{T}(\sigma)$.

Observe that $\mathcal{B}(S) = \mathcal{B}(\delta S)$ for any scalar δ. Thus, $\mathcal{B}(T(A, b, c)) = \mathcal{B}(\tilde{T}(A, b, c))$.

This Lemma is proved in the Appendix:

Lemma 4.2: The complement $\mathcal{G}_0^c$ of $\mathcal{G}_0$ is a proper algebraic subset of $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$.

The key property of this set is as follows.

Lemma 4.3: For each $(A, N, b, c) \in \mathcal{G}_0$, consider $M := TN'T^{-1}$, where $T = T(A, b, c)$. Then,

1) $M \neq N$,
2) $(A, M, b, c) \in \mathcal{G}_0$, and
3) for each $\gamma \in \mathbb{R}$ and each nonnegative integer k:

$$c(A + \gamma N)^k b = c(A + \gamma M)^k b. \tag{14}$$

Proof: Let $(A, N, b, c) \in \mathcal{G}_0$. The fact that $M \neq N$ follows from the fact that $N \notin \mathcal{B}(T(A, b, c))$. To see that $(A, M, b, c) \in \mathcal{G}_0$, note that $MT = TN'T^{-1}T = TN' \neq TM'$ since $N \neq M$ and T is invertible. The equality (14) follows by the equalities $AT = TA', b = Tc', cT = b', TN' = MT$, and the following:

$$\begin{aligned} c(A + \gamma N)^k b &= \left(c(A + \gamma N)^k b\right)' = b'(A' + \gamma N')^k c' \\ &= cT(T^{-1}AT + \gamma T^{-1}MT)^k T^{-1}b \\ &= c(A + \gamma M)^k b \end{aligned}$$

for all $k \geq 0$ and all $\gamma \in \mathbb{R}$. ∎

Corollary 4.4: For each $(A, N, b, c) \in \mathcal{G}_0$, and $M = TN'T^{-1}$, the 4-tuples (A, N, b, c) and (A, M, b, c) are not i/o equivalent to each other.

Proof: Suppose that these two 4-tuples would be i/o equivalent. By Lemma 2.3, they are similar. Let T provide a similarity as in (6). In particular, T provides a similarity between the canonical triple (A, b, c) and itself. Since there is a unique such similarity, and the identity I is one, it follows that $T = I$. Thus $N = TMT^{-1} = M$, contradicting the fact that $N \neq M$. ∎

B. Proof of Theorem 1

Let $\mathcal{C}$ be the subset consisting of all those 4-tuples

$$(Q, N, b_0, c) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$$

which satisfy the following conditions:

a) (Q, b_0, c) is canonical;
b) $(Q - N, b_0, c)$ is canonical;
c) $e^Q - I$ is invertible;
d) $N \notin \mathcal{B}(T(Q, b_0, c))$.

Letting $\Lambda(F)$ denote the collection of eigenvalues of a matrix F, the Spectral Mapping Theorem implies that $e^{\Lambda(Q)} = \{e^\lambda : \lambda \in \Lambda(Q)\}$. Thus, assumption (c), which says that 1 is not an eigenvalue of e^Q, implies, in particular, that Q is invertible. This Lemma (see Appendix for proof) will help us show genericity of the set of systems to be considered:

Lemma 4.5: The complement $\mathcal{C}^c$ of $\mathcal{C}$ is a countable union $\mathcal{C}^c = \bigcup_{k=0}^{\infty} \mathcal{E}_k$, where each $\mathcal{E}_k$ is a proper algebraic subset of $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$. □

Let $\mathcal{X} = \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$, and consider the analytic map $\psi : \mathcal{X} \to \mathcal{X}$ defined by

$$\psi : (Q, N, b_0, c) \mapsto \left(Q - N, N, (\rho(Q))^* b_0, c\right)$$

where $\rho(Q) = \int_0^1 e^{sQ} ds$, and $\rho(Q)^*$ denotes the adjoint matrix of $\rho(Q)$. Note that if $e^Q - I$ is invertible, then Q is invertible, and the matrix $\rho(Q) = Q^{-1}(e^Q - I)$ is also invertible. Hence, when restricted to the open set

$$\mathcal{X}_1 := \left\{(Q, N, b_0, c) : e^Q - I \text{ is invertible}\right\}$$

ψ is given by

$$(Q, N, b_0, c) \mapsto (Q - N, N, \kappa(b_0), c) \qquad (15)$$

where $\kappa(b_0) = \det(\rho(Q))(\rho(Q))^{-1}b_0$. Let ψ_0 denote the restriction of ψ to $\mathcal{X}_1$, and consider the open set

$$\mathcal{Y}_1 := \left\{ (A, N, b, c) : (e^{A+N} - I) \text{ is invertible} \right\}.$$

Then, ψ_0 is a (smooth) diffeomorphism from $\mathcal{X}_1$ to $\mathcal{Y}_1$. Its inverse is $\psi_0^{-1} : \mathcal{Y}_1 \to \mathcal{X}_1$ given by

$$\psi_0^{-1} : (A, N, b, c) \mapsto \left(A + N, N, \left[\det \left(\int_0^1 e^{s(A+N)} ds \right) \right]^{-1} \right.$$
$$\left. \left(\int_0^1 e^{s(A+N)} ds \right) b, c \right). \qquad (16)$$

Since $\mathcal{Y} = \psi(\mathcal{X}_1) \subseteq \psi(\mathcal{X})$, it follows that the complement $(\psi(\mathcal{X}))^c$ of $\psi(\mathcal{X})$ is a subset of $\{(A, N, b, c) : (e^{A+N} - I) \text{ is singular}\}$. Hence, $(\psi(\mathcal{X}))^c$ is contained in the countable union of the solution sets $\mathcal{F}_k$ of the equations

$$\det \left((A + N)^2 + 4k^2\pi^2 I \right) = 0, \quad k = 0, 1, 2, \ldots.$$

Let $\mathcal{D} = \psi(\mathcal{C})$, and write $\mathcal{C}^c = \bigcup_{k=0}^{\infty} \mathcal{E}_k$, where the sets $\mathcal{E}_k$ are as in Lemma 4.5. The next lemma then follows from the fact that $\mathcal{D}^c \subseteq [\psi(\mathcal{X})]^c \bigcup \psi(\mathcal{C}^c)$, and Lemma 4.5:

Lemma 4.6: $\mathcal{D}^c \subseteq (\bigcup_{k=0}^{\infty} \psi_0(\mathcal{E}_k)) \bigcup (\bigcup_{k=0}^{\infty} \mathcal{F}_k)$, where ψ_0 is the diffeomorphism from $\mathcal{X}_1$ to $\mathcal{Y}_1$ defined by (15). $\quad \square$

Corollary 4.7: $\mathcal{D}$ is generic.

Proof: Since every proper algebraic set has measure zero, the set $\bigcup_{k \geq 1} \mathcal{F}_k$ has measure zero. Furthermore, since the image of a measure zero set under a differentiable map has measure zero (see e.g. [23], Lemma 2.6), $\psi(\mathcal{E}_k)$ has measure zero for each k. This implies that $\mathcal{D}^c$ has measure zero. Therefore, $\mathcal{D}$ is of full measure, and as a consequence, $\mathcal{D}$ is dense. Finally, $\mathcal{D}$ is open because it is the image of $\mathcal{C}$ under the diffeomorphism ψ_0 and $\mathcal{Y}_1$ is an open subset of $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$. $\quad \blacksquare$

Let $\mathbf{u} = \mathbf{u}_{\tau,\alpha}$ with $\tau = 1, \alpha = 1$.

Lemma 4.8: Consider systems as in (4). For every $\sigma^o \in \mathcal{D}$, there exists $\widehat{\sigma^o} \in \mathcal{D}$ such that the following holds:
1) σ^o and $\widehat{\sigma^o}$ are i/o equivalent under the pulse function $\mathbf{u}$, but
2) σ^o and $\widehat{\sigma^o}$ are not i/o equivalent.

Proof: Let $\sigma^o = (A, N, b, c) \in \mathcal{D} = \psi(\mathcal{C})$. Then there exists $(Q, N, b_0, c) \in \mathcal{C} \subseteq \mathcal{G}_0$ such that

$$A = Q - N, \quad b = [\det(\rho(Q))](\rho(Q))^{-1} b_0.$$

Let $b_1 = \rho(Q)b$. Then $b_1 = \det(\rho(Q))b_0$, and hence,

$$\mathcal{R}(Q, b_1) = \det(\rho(Q)) \, \mathcal{R}(Q, b_0)$$
$$\mathcal{R}(Q - N, b_1) = \det(\rho(Q)) \, \mathcal{R}(Q - N, b_0).$$

Since $\det(\rho(Q)) \neq 0$, both (Q, b_1) and $(Q - N, b_1)$ are reachable. Moreover,

$$T(Q, b_1, c) = T(Q, \det(\rho(Q)) b_0, c)$$
$$= \mathcal{R}(Q, \det(\rho(Q)) b_0) \left[\mathcal{O}(Q, c)' \right]^{-1}$$
$$= \det(\rho(Q)) \mathcal{R}(Q, b_0) \left[\mathcal{O}(Q, c)' \right]^{-1}$$
$$= \det(\rho(Q)) T(Q, b_0, c)$$

which implies that $\mathcal{B}(T(Q, b_1, c)) = \mathcal{B}(T(Q, b_0, c))$. Therefore, $(Q, N, b_1, c) \in \mathcal{C}$. In particular, $(Q, N, b, c) \in \mathcal{G}_0$.

Applying Lemma 4.3 to (Q, N, b_1, c), one sees that with $M = TN'T^{-1}$ (where $T = T(Q, b_1, c)$), it holds that $M \neq N$, and

$$ce^{t(Q+\gamma N)}b_1 = ce^{t(Q+\gamma M)}b_1 \quad \forall \gamma, \forall t \geq 0.$$

In particular, for $\gamma = -1$,

$$ce^{t(Q-N)}b_1 = ce^{t(Q-M)}b_1 \qquad \forall t \geq 0.$$

Let $\widehat{\sigma^o} = ((A + N - M), M, b, c)$. Consider the two systems σ^o and $\widehat{\sigma^o}$:

$$\dot{x} = (A + uN)x + bu, \quad x(0) = 0, \quad y = cx$$
$$\dot{z} = [(A + N - M) + uM]z + bu, \quad z(0) = 0, \quad y = cz.$$

Since $\mathbf{u} = 1$ for $t \in [0, 1]$, the two systems reduce in that interval to:

$$\dot{x} = (A + N)x + bu, \quad x(0) = 0, \quad y = cx, \quad 0 \leq t \leq 1$$
$$\dot{z} = (A + N)z + bu, \quad z(0) = 0, \quad y = cz, \quad 0 \leq t \leq 1.$$

It follows that $x(t) = z(t)$ on [0, 1], and hence outputs coincide for $t \in [0, 1]$. In particular, at time $t = 1$ both systems are in state

$$\int_0^1 e^{(1-s)(A+N)}bds = \int_0^1 e^{(1-s)Q}bds = \rho(Q)b = b_1.$$

Now, for $t \geq 1$, using that $A = Q - N$ and $A + N - M = Q - M$, we have that

$$\dot{x} = (Q - N)x, \quad x(1) = b_1$$
$$\dot{z} = (Q - M)z, \quad z(1) = b_1$$

and, since $ce^{t(Q-N)}b_1 \equiv ce^{t(Q-M)}b_1$, it follows that the outputs are the same for all $t > 1$ as well.

To show that σ^o and $\widehat{\sigma^o}$ are not equivalent, we will show explicitly that the inputs

$$u_s(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & 1 < s \leq 1 + s \\ 1, & t > 1 + s \end{cases}$$

(with varying $s \geq 0$) are enough to distinguish the two systems.

Suppose that the systems have the same output functions under these input functions.

Let $s \geq 0$ and $t \geq 0$. Since $x(1) = z(1) = b_1$, and $x(1+s) = e^{s(Q-N)}b_1$, $z(1+s) = e^{s(Q-M)}b_1$, at time $t > 1 + s$ we have:

$$\dot{x} = Qx + b, \quad x(1+s) = e^{s(Q-N)}b_1$$
$$\dot{z} = Qz + b, \quad z(1+s) = e^{s(Q-M)}b_1.$$

From $cx(t) \equiv cz(t)$ and taking derivatives, one gets

$$cQx(t) + cb = cQz(t) + cb \qquad \forall t > 1 + s.$$

Hence,

$$cQx(t) = cQz(t) \qquad \forall t > 1 + s.$$

Taking more derivatives, we conclude inductively that

$$cQ^k x(t) = cQ^k z(t) \qquad \forall t > 1 + s, \forall k \geq 0.$$

In particular by continuity, we have that

$$cQ^k x(1+s) = cQ^k z(1+s)$$

for all $k \geq 0$, and by observability of (Q, c),

$$x(1+s) = z(1+s) \qquad \forall s \geq 0$$

that is,

$$e^{s(Q-N)}b_1 = e^{s(Q-M)}b_1 \qquad \forall s \geq 0.$$

Taking derivatives with respect to s, we conclude that:

$$(Q - N)^k b_1 = (Q - M)^k b_1 \qquad \forall k \geq 0.$$

This implies that

$$(Q - N)R_1 = (Q - M)R_2, \quad \text{and} \quad R_1 = R_2$$

where $R_1 = \mathcal{R}(Q - N, b_1)$, and $R_2 = \mathcal{R}(Q - M, b_1)$. Since $(Q-N, b_1)$ is reachable, R_1 is invertible. From this we conclude that $Q - N = Q - M$, and hence $N = M$, a contradiction.

To complete the proof of Lemma 4.8, we show that $\widehat{\sigma^o} \in \mathcal{D}$. First observe that:

$$\widehat{\sigma^o} = (A + N - M, M, b, c) = (Q - M, M, b, c)$$
$$= \psi_0(Q, M, b_0, c).$$

Thus, if we prove that $(Q, M, b_0, c) \in \mathcal{C}$, then $\widehat{\sigma^o} = \psi(Q, M, b_0, c) \in \mathcal{D}$. By Lemma 4.3, $(Q, M, b_0, c) \in \mathcal{G}_0$. It is thus enough to show that $(Q - M, b_0, c)$ is canonical. To see this, note that since $b_1 = \det(\rho(Q))b_0$ and $((Q - N), b_0, c)$ is canonical, it follows that $(Q - N, b_1, c)$ is canonical. Thus, $(Q - M, b_1, c)$ is canonical as $(Q - M, b_1, c)$ is similar to $((Q - N)', c', b_1')$. Again, applying the fact that $b_1 = \det(\rho(Q))b_0$, one sees that $(Q - M, b_0, c)$ is canonical. ∎

The above completes the proof of Theorem 1 for the special case $\tau = 1$ and $\alpha = 1$. The general case can be obtained by rescaling inputs and time scale, as follows. We consider the impulse function $\mathbf{u}_{\tau,\alpha}$ for any fixed $\tau > 0$ and $\alpha \in \mathbb{R}$. Without loss of generality, we assume that $\alpha \neq 0$. For the initial-value problem

$$\dot{x} = (A + \alpha N)x + \alpha b, x(0) = 0$$

let $N_\alpha = \alpha N, b_\alpha = \alpha b$, and consider the initial-value problem

$$\dot{\tilde{x}} = \frac{1}{\tau}\left((A + N_\alpha)\tilde{x} + b_\alpha\right), \quad \tilde{x}(0) = 0.$$

Then $\tilde{x}(t) = x(t/\tau)$. It then can be seen that, with

$$\mathcal{D}_{\alpha,\tau} := \left\{ \left(\frac{1}{\tau}A, \frac{\alpha}{\tau}N, \frac{\alpha}{\tau}b, c\right) : (A, N, b, c) \in \mathcal{D} \right\}$$

and any $\sigma \in \mathcal{D}_{\alpha,\tau}$, there exists some $\widehat{\sigma} \in \mathcal{D}_{\alpha,\tau}$ such that σ and $\widehat{\sigma}$ have the same output under the impulse input $\mathbf{u}_{\tau,\alpha}$, but the two systems are not equivalent.

C. Proof of Theorem 2

In this section we consider systems defined as in (5). Let $\tau \geq 0$ and $\alpha \in \mathbb{R}$ be given. Consider the analytic map Φ:

$$\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n} \to \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$$

given by

$$(P, N, b_0, c) \mapsto (P - \alpha N, N, e^{-\tau P}b_0, c).$$

This is an analytic diffeomorphism whose inverse map is given by

$$(A, N, b, c) \mapsto \left(A + \alpha N, N, e^{\tau(A+\alpha N)}b, c\right).$$

Let $\mathcal{H} = \Phi(\mathcal{G}_0)$. The following is a consequence of Lemma 4.2:

The complement $\mathcal{H}^c$ of $\mathcal{H}$ is the image of an proper algebraic set under a diffeomorphism from $\mathcal{X}$ to $\mathcal{X} : \mathcal{H}^c = \Phi(\mathcal{G}_0^c)$. Since the image of a measure zero set under a smooth map (see e.g. [23], Lemma 2.6) has measure zero, we conclude:

Corollary 4.9: The collection $\mathcal{H}$ is generic.   □

It can be seen that to prove Theorem 2, it is enough to prove the following:

Lemma 4.10: For any $\sigma_1 \in \mathcal{H}$, there exists $\sigma_2 \in \mathcal{H}$ such that
 a) σ_1 and σ_2 have the same output for any $u \in \mathcal{I}_{\tau,\alpha}$;
 b) the two systems σ_1 and σ_2 are not equivalent.
Let $\sigma_1 = (A, N, b, c) \in \mathcal{H}$. Thus, by definition of $\mathcal{H}$, there exists $(P, N, b_0, c) \in \mathcal{G}_0$ such that $\sigma_1 = (P - \alpha N, N, e^{-\tau P}b_0, c)$. Let $M = TN'T^{-1}$, where $T = T(P, b_0, c)$. Then $M \neq N$, $(P, M, b_0, c) \in \mathcal{G}_0$, and the system $\sigma_2 := (P - \alpha M, M, e^{-\tau P}b_0, c) = \Phi(P, M, b_0, c) \in \mathcal{H}$.

To prove part (a), pick any $u \in \mathcal{I}_{\tau,\alpha}$, and assume $u(t) = \beta$ for $t > \tau$. The two systems are given by

$$\dot{x} = ((P - \alpha N) + uN)x, \quad x(0) = b$$
$$\dot{z} = ((P - \alpha M) + uM)z, \quad z(0) = b.$$

For $0 \leq t \leq \tau$, both systems reduce to the same equation:

$$\dot{p} = Pp, \quad p(0) = b = e^{-\tau P}b_0$$

and in particular, $cx(t) = cz(t) = ce^{tP}b$ for all $0 \leq t \leq \tau$. For $t > \tau$, the two systems become

$$\dot{x} = ((P - \alpha N) + \beta N) x, \quad x(\tau) = b_0$$
$$\dot{z} = ((P - \alpha M) + \beta M) z, \quad z(\tau) = b_0$$

and thus have the respective solutions

$$x(t) = ce^{(P+(\beta-\alpha)N)(t-\tau)}x(\tau) = ce^{(P+(\beta-\alpha)N)(t-\tau)}b_0$$
$$z(t) = ce^{(P+(\beta-\alpha)M)(t-\tau)}z(\tau) = ce^{(P+(\beta-\alpha)M)(t-\tau)}b_0.$$

By Lemma 4.3 and the choices of M and N, it follows with $\gamma = \beta - \alpha$ that:

$$ce^{(P+\gamma N)(t-\tau)}b_0 = ce^{(P+\gamma M)(t-\tau)}b_0$$

which implies that $cx(t) = cz(t)$ for all $t \geq \tau$ as well, so part (a) is proved.

To prove part (b), suppose the two systems σ_1 and σ_2 have the same output for all inputs, and so, in particular, for all inputs u for which $u(t) = \alpha$ for $0 \leq t \leq \tau$. This implies that the two systems

$$\dot{x} = ((P - \alpha N) + uN) x, \quad x(\tau) = e^{\tau P}b, \quad y = cx$$
$$\dot{z} = ((P - \alpha M) + uM) z, \quad z(\tau) = e^{\tau P}b, \quad y = cz$$

have the same output for any input. Rewriting the two systems as:

$$\dot{x} = (P + (u - \alpha)N) x, \quad x(\tau) = e^{\tau P}b, \quad y = cx$$
$$\dot{z} = (P + (u - \alpha)M) z, \quad z(\tau) = e^{\tau P}b, \quad y = cz$$

and writing $v(t) = u(t) - \alpha$ for a new input v, one sees that the two systems

$$\dot{x} = (P + vN) x, \quad x(\tau) = b_0, \quad y = cx$$
$$\dot{z} = (P + vM) z, \quad z(\tau) = b_0, \quad y = cz$$

have the same outputs for all inputs v and times $\geq \tau$, which is the same as saying that the two systems of type II with associated 4-tuples (P, N, b_0, c) and (P, M, b_0, c) are i/o equivalent, which is a contradiction in view of Lemma 4.3 and Corollary 4.4.

This completes the proof of Theorem 2.

V. PROOFS OF POSITIVE RESULTS

In this section we prove the positive results.

Let $\alpha \neq 0$ be given. Let $\mathcal{M}$ be the set of 4-tuples satisfying the following two properties:
1) (A, b, c) is canonical,
2) $(A + \alpha N, b)$ is controllable.
Since the complement $\mathcal{M}^c$ of $\mathcal{M}$ is defined by the union of the solution sets of the equations

$$\det \mathcal{R}(A, b) = 0$$
$$\det \mathcal{O}(A, c) = 0$$
$$\det \mathcal{R}(A + \alpha N, b) = 0$$

we have:

Lemma 5.1: The complement $\mathcal{M}^c$ of $\mathcal{M}$ is a proper algebraic set. Consequently, $\mathcal{M}$ is open, dense, and of full measure.

We will prove that the sets of systems in $\mathcal{S}_n^I$ and $\mathcal{S}_n^{II}$ whose 4-tuples are in $\mathcal{M}$ satisfy the conclusions of Theorems 3 and 4 respectively.

A. Proof of Theorem 3

Lemma 5.2: Assume that (A, b) is controllable. Then, for any $N \in \mathbb{R}^{n \times n}$ and any $\alpha \in \mathbb{R}$,

$$\left(A, \left(\int_0^\tau e^{(A+\alpha N)s}ds \right) b \right)$$

is controllable for almost all $\tau > 0$.

Proof: Let A, N, b be given. For any $\tau > 0$,

$$\int_0^\tau e^{(A+\alpha N)s}ds = \sum_{k=0}^\infty \frac{(A+\alpha N)^k}{(k+1)!}\tau^{k+1} = \tau\Psi(\tau)$$

where $\Psi(\tau) = \sum_{k=0}^\infty ((A + \alpha N)^k/(k + 1)!)\tau^k$. Since $\Psi(0) = I$, Ψ is analytic, and (A, b) is controllable, it follows that $(A, \Psi(\tau)b)$ is controllable for almost all τ. It then follows that $(A, \tau\Psi(\tau)b)$ is controllable for almost all $\tau > 0$. ∎

To prove Theorem 3, we pick two systems $\sigma_1^o = (A_1, N_1, b_1, c_1)$ and $\sigma_2^o = (A_2, N_2, b_2, c_2)$ in $\mathcal{M}$, and suppose that they produce have the same output function for each $u \in \mathcal{V}_\alpha$. We must show that $\sigma_1^o \equiv \sigma_2^o$.

Fix a $\tau > 0$. Applying $\mathbf{u}_{\tau,\alpha} \in \mathcal{V}_\alpha$ to the two systems:

$$\dot{x} = (A_1 + uN_1)x + b_1 u, \quad x(0) = 0, \quad y = c_1 x$$
$$\dot{z} = (A_2 + uN_2)z + b_2 u, \quad z(0) = 0, \quad y = c_2 z$$

one has:

$$c_1 e^{A_1(t-\tau)}x(\tau) = c_2 e^{A_2(t-\tau)}z(\tau) \qquad \forall t \geq \tau \qquad (17)$$

where

$$x(\tau) = \alpha \int_0^\tau e^{(A_1+\alpha N_1)(\tau-s)}ds b_1$$

$$= \alpha \int_0^\tau e^{(A_1+\alpha N_1)s}ds b_1$$

$$z(\tau) = \alpha \int_0^\tau e^{(A_2+\alpha N_2)(\tau-s)}ds b_2$$

$$= \alpha \int_0^\tau e^{(A_2+\alpha N_2)s}ds b_2.$$

This holds for any $\tau > 0$.

By Lemma 5.2, we may pick some $\tau_0 > 0$ such that $(A_1, x(\tau_0), c_1)$ and $(A_2, z(\tau_0), c_2)$ are both canonical. Since by (17) these two triples are i/o equivalent, there exists some invertible matrix $T \in \mathbb{R}^{n \times n}$ such that

$$A_2 = T^{-1}A_1 T, \quad z(\tau_0) = T^{-1}x(\tau_0), \quad c_2 = c_1 T. \qquad (18)$$

Using in particular that $c_2 e^{A_2 s} = c_1 e^{A_1 s} T$ for all s, (17) becomes:

$$c_1 e^{A_1 (t-\tau)} x(\tau) = c_1 e^{A_1 (t-\tau)} T z(\tau) \qquad \forall t \geq \tau.$$

From the observability of (A_1, c_1), it follows that:

$$x(\tau) = T z(\tau)$$

for all $\tau > 0$. Equivalently:

$$\int_0^\tau e^{(A_1 + \alpha N_1)s} ds\, b_1 = T \int_0^\tau e^{(A_2 + \alpha N_2)s} ds\, b_2$$

for all $\tau > 0$. Taking the derivative with respect to τ, one gets:

$$e^{(A_1 + \alpha N_1)\tau} b_1 = T e^{(A_2 + \alpha N_2)\tau} b_2.$$

Note that this is true for all $\tau \geq 0$. In particular,

$$b_1 = T b_2. \tag{19}$$

On the other hand, taking repeated derivatives in τ and then setting $\tau = 0$, one obtains:

$$(A_1 + \alpha N_1)^k b_1 = T(A_2 + \alpha N_2)^k b_2 \qquad \forall k \geq 0. \tag{20}$$

This implies, with $0 \leq k \leq n - 1$,

$$\mathcal{R}(A_1 + \alpha N_1, b_1) = T\left[\mathcal{R}(A_2 + \alpha N_2, b_2)\right] \tag{21}$$

and with $1 \leq k \leq n$,

$$(A_1 + \alpha N_1)\left[\mathcal{R}(A_1 + \alpha N_1, b_1)\right]$$
$$= T(A_2 + \alpha N_2)\left[\mathcal{R}(A_2 + \alpha N_2, b_2)\right].$$

Combining this with (21), one sees that

$$(A_1 + \alpha N_1)\left[\mathcal{R}(A_1 + \alpha N_1, b_1)\right]$$
$$= T(A_2 + \alpha N_2)T^{-1}\left[\mathcal{R}(A_1 + \alpha N_1, b_1)\right]$$

It then follows from the fact that $\mathcal{R}(A_1 + \alpha N_1, b_1)$ is invertible (because $(A_1 + \alpha N_1, b_1)$ is controllable) that:

$$T(A_2 + \alpha N_2)T^{-1} = (A_1 + \alpha N_1).$$

It then again follows from (18) and the fact that $\alpha \neq 0$ that $N_2 = T^{-1} N_1 T$. Combined with (18) and (19), we have that the systems σ_1^o and σ_2^o are similar, with the similarity matrix given by T, and this completes the proof of Theorem 3.

B. Proof of Theorem 4

The proof of Theorem 4 is almost the same as that of Theorem 3, with Lemma 5.2 modified to the following:

Lemma 5.3: Assume that (A, b) is controllable. Then, for any $N \in \mathbb{R}^{n \times n}$ and any $\alpha \in \mathbb{R}$,

$$\left(A, e^{(A + \alpha N)\tau} b\right)$$

is controllable for almost all $\tau > 0$. $\square$

We pick elements $\sigma_1 = (A_1, N_1, b_1, c_1)$ and $\sigma_2 = (A_2, N_2, b_2, c_2)$ of $\mathcal{M}$ (seen as a class of systems of type II) which have the same output function for each $u \in \mathcal{V}_\alpha$.

For any $\tau > 0$, applying $\mathbf{u}_{\tau,\alpha} \in \mathcal{V}_\alpha$ to the two systems:

$$\dot{x} = (A_1 + uN_1)x, \quad x(0) = b_1, \quad y = c_1 x$$
$$\dot{z} = (A_2 + uN_2)z, \quad z(0) = b_2, \quad y = c_2 z$$

one has, for any $\tau > 0$

$$c_1 e^{A_1(t-\tau)} x(\tau) = c_2 e^{A_2(t-\tau)} z(\tau) \qquad \forall t \geq \tau \tag{22}$$

where

$$x(\tau) = e^{(A_1 + \alpha N_1)\tau} b_1, \quad z(\tau) = e^{(A_2 + \alpha N_2)\tau} b_2.$$

By Lemma 5.3, there exists some $\tau_0 > 0$ such that both $(A_1, x(\tau_0), c_1)$ and $(A_2, z(\tau_0), c_2)$ are canonical. So, there exists some invertible matrix $T \in \mathbb{R}^{n \times n}$ such that

$$A_2 = T^{-1} A_1 T, \quad z(\tau_0) = T^{-1} x(\tau_0), \quad c_2 = c_1 T$$

and consequently, (22) becomes

$$c_1 e^{A_1(t-\tau)} x(\tau) = c_1 e^{A_1(t-\tau)} T z(\tau) \qquad \forall t \geq \tau.$$

For each $\tau > 0$ given, using the observability of (A_1, c_1), one sees that:

$$e^{(A_1 + \alpha N_1)\tau} b_1 = T e^{(A_2 + \alpha N_2)\tau} b_2$$

for all $\tau > 0$. Starting from here, one can complete the proof by following the same steps as in the proof of Theorem 3.

VI. REMARK ON SAMPLED CONTROLS

As remarked earlier, our proofs of the positive results, Theorems 3 and 4, were inspired by the identification algorithm presented in [12]. That algorithm aims to find a system equivalent to the system being identified, on the basis of observations at discrete instants $0, \tau, 2\tau, \ldots, j\tau, \ldots$, where τ is a *fixed* sampling time, and having applied inputs which have the form $\mathbf{u}_{k\tau,\alpha}$ (for varying nonnegative integer k's), i.e., pulses of magnitude α whose width is a multiple of this same sampling time τ. The motivation is clear: one wishes to use a sample-and-hold strategy, which is especially convenient for computer algorithms. Unfortunately, this restriction to fixed sampling times means that the algorithm cannot work for generic classes of systems, as we show here by means of a counterexample. (Mathematically, the difficulty is that some of the steps of the algorithm given in [12] involve taking logarithms of matrices, which is an ambiguous procedure, as the author himself points out on the paper.) To show this shortcoming, for any given $\alpha > 0$, we produce an open class $\mathcal{B}_\alpha$ of 2-dimensional systems of type I (it is easy to generalize to larger dimensions and to systems of type II) with the following properties: for every system in $\mathcal{B}_\alpha$, there is some other system, which is not i/o equivalent to the original one, yet cannot be distinguished by applying steps of magnitude α and sampled in the above way (with fixed τ). Thus, our approach, in which τ is varied, is actually necessary.

For any system $\sigma^o = (A, N, b, c) \in \mathcal{S}_n^I$, we denote by σ_τ^o the discrete time system which results from sampling the system with τ as the length of the sampling interval, and using input functions u that are constant over each sampling interval:

$$x_{k+1} = F_k x_k + u_k g_k, \quad x(0) = 0, \quad y_k = c x_k \quad (23)$$

where $x_k = x(k\tau)$, $y_k = y(k\tau)$, u_k is the value of u over the interval $(k\tau, (k+1)\tau)$ and

$$F_k = e^{(A + u_k N)\tau}$$

$$g_k = \int_0^\tau e^{(A + u_k N)(\tau - s)} b \, ds.$$

For disrete time systems as in (23), the i/o equivalence under a collection of inputs is defined in the same manner as in the continuous time case.

Let $\mathcal{B}_\alpha$ be the subset consisting of systems of type I for which the 4-tuples

$$(A, N, b, c) \in \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 1} \times \mathbb{R}^{1 \times 2}$$

satisfy:
1) $(A + \alpha N, b, c)$ is canonical; and
2) $A + \alpha N$ has a pair of conjugate complex eigenvalues $r \pm si$, with $s \neq 0$.

Since the set of 4-tuples for which the triple $(A + \alpha N, b, c)$ is canonical is generic, and the set of 4-tuples (A, N, b, c) for which $A + \alpha N$ has a pair of nonzero conjugate complex eigenvalues has a nonempty interior (because of continuity of eigenvalues on matrix entries), the set $\mathcal{B}_\alpha$ contains an open set.

Proposition 6.1: For any $\tau > 0$, and for any $\sigma^o \in \mathcal{B}_\alpha$, there exists some $\widehat{\sigma^o} \in \mathcal{B}_\alpha$ that
1) σ_τ^o and $\widehat{\sigma^o}_\tau$ are i/o equivalent under the collection $\{\mathbf{u}_{k\tau,\alpha}\}_{k \geq 0}$, but
2) σ^o and $\widehat{\sigma^o}$ are not i/o equivalent.

Proof: Let $\tau > 0$ be given, and consider $\sigma^o = (A, N, b, c) \in \mathcal{B}_\alpha$. Without loss of generality, assume that $A + \alpha N$ is already in "real Jordan canonical form":

$$A + \alpha N = \begin{pmatrix} r & -s \\ s & r \end{pmatrix}$$

for some $s \neq 0$. (If this were not the case, one may simply apply a similarity, and at the end of the argument transform back to original coordinates.) Let

$$\Lambda_0 = \begin{pmatrix} 0 & -2\pi/\tau \\ 2\pi/\tau & 0 \end{pmatrix}.$$

Choose an integer $l \neq 0$ such that
- $s + (2l\pi/\tau) \neq 0$; and
- $((A + \alpha N) + l\Lambda_0, b, c)$ is canonical.

A generic integer works; note that (since $n = 2$):

$$\det \mathcal{O}(A + \alpha N + l\Lambda_0, c)$$
$$= \det \mathcal{O}(A + \alpha N, c) + \det \mathcal{O}(l\Lambda_0, c)$$
$$= \det \mathcal{O}(A + \alpha N, c) + l \det \mathcal{O}(\Lambda_0, c).$$

and similarly for reachability. Let M be given by

$$M = N + \frac{l}{\alpha} \Lambda_0.$$

That is, M is chosen so that

$$A + \alpha M = A + \alpha N + l\Lambda_0 = \begin{pmatrix} r & -s - \frac{2l\pi}{\tau} \\ s + \frac{2l\pi}{\tau} & r \end{pmatrix}.$$

Note that both $(A + \alpha N)$ and $(A + \alpha M)$ are invertible since $r + si \neq 0$ and $r + (s + (2l\pi/\tau))i \neq 0$. Let $\widehat{\sigma^o} = (A, M, \hat{b}, c)$, where $\hat{b} = (A + \alpha M)(A + \alpha N)^{-1} b$.

Next we show that the two sampled systems σ_τ^o and $\widehat{\sigma^o}_\tau$ are i/o equivalent under the collection $\{\mathbf{u}_{k\tau,\alpha}\}_{k \geq 0}$.

Consider the input function $\mathbf{u}_{k\tau,\alpha}$ for some $k \geq 0$. Clearly the two systems have the same state trajectory and same output function if $k = 0$ (the input is constantly zero). So assume $k \geq 1$. For $j \leq k - 1$, the two sampled systems are given respectively by:

$$x_{j+1} = F x_j + \alpha g, \quad x_0 = 0, \quad y_j = c x_j$$
$$z_{j+1} = \widehat{F} z_j + \alpha \widehat{g}, \quad z_0 = 0, \quad y_j = c z_j$$

where

$$F = e^{(A + \alpha N)\tau} = e^{r\tau} \begin{pmatrix} \cos s\tau & -\sin s\tau \\ \sin s\tau & \cos s\tau \end{pmatrix}$$
$$= e^{r\tau} \begin{pmatrix} \cos \left(s + \frac{2l\pi}{\tau}\right)\tau & -\sin \left(s + \frac{2l\pi}{\tau}\right)\tau \\ \sin \left(s + \frac{2l\pi}{\tau}\right)\tau & \cos \left(s + \frac{2l\pi}{\tau}\right)\tau \end{pmatrix}$$
$$= e^{(A + \alpha M)\tau} = \widehat{F}$$

and

$$g = \int_0^\tau e^{(A + \alpha N)(\tau - \theta)} b \, d\theta$$
$$= \left(e^{(A + \alpha N)\tau} - I\right)(A + \alpha N)^{-1} b$$
$$= \left(e^{(A + \alpha M)\tau} - I\right)(A + \alpha M)^{-1} \hat{b}$$
$$= \int_0^\tau e^{(A + \alpha M)(\tau - \theta)} \hat{b} \, d\theta = \widehat{g}$$

(where we have used the fact that $\int_0^\tau e^{Q(\tau - \theta)} d\theta = (e^{Q\tau} - I)Q^{-1}$ for any invertible Q). Hence, $x_j = z_j$ for all $0 \leq j \leq k$. In particular, outputs coincide at all sampling times $t = j\tau$, $j \leq k$. Over the interval $[k\tau, \infty)$, the two systems σ^o and $\widehat{\sigma^o}$ are given by

$$\dot{w} = Aw, \quad w(k\tau) = x(k\tau) = z(k\tau), \quad y = cw.$$

This in particular implies that $x_j = z_j$ for all $j \geq k$ and therefore outputs also coincide at sampling times $t = j\tau$, $j > k$ (as well as for any time $t > k$). Thus, σ_τ^o and $\widehat{\sigma^o}_\tau$ are i/o equivalent under $\mathbf{u}_{k\tau,\alpha}$.

To show that the two systems σ^o and $\widehat{\sigma^o}$ are not i/o equivalent, consider the constant input $u \equiv \alpha$. Assume the output functions of the two systems are the same under this constant u. Then

$$c \int_0^t e^{(A+\alpha N)(t-\theta)} b\alpha d\theta$$
$$= c \int_0^t e^{(A+\alpha M)(t-\theta)} \widehat{b}\alpha d\theta \quad \forall t \geq 0$$

and equivalently,

$$c \left(e^{(A+\alpha N)t} - I \right)(A+\alpha N)^{-1}b$$
$$= c \left(e^{(A+\alpha M)t} - I \right)(A+\alpha M)^{-1}\widehat{b} \quad \forall t \geq 0.$$

Since $(A+\alpha M)^{-1}\widehat{b} = (A+\alpha N)^{-1}b$, one gets

$$c \left(e^{(A+\alpha N)t} - I \right)(A+\alpha N)^{-1}b$$
$$= c \left(e^{(A+\alpha M)t} - I \right)(A+\alpha N)^{-1}b \quad \forall t \geq 0.$$

This in turn implies that

$$\int_0^t c e^{(A+\alpha N)(t-\theta)} b d\theta = \int_0^t c e^{(A+\alpha M)(t-\theta)} b d\theta \quad \forall t \geq 0.$$

Taking derivatives with respect to t repeatedly, and then setting $t = 0$, one obtains:

$$c(A+\alpha N)^j b = c(A+\alpha M)^j b \quad \forall j \geq 0.$$

Since both $(A+\alpha N, b, c)$ and $(A+\alpha M, b, c)$ are canonical, it follows that the two systems are similar, contradicting the fact that $A+\alpha N$ and $A+\alpha M$ have different pairs of eigenvalues.

Finally we show that $\widehat{\sigma^o} \in \mathcal{B}_\alpha$. Note that M was chosen so that $(A+\alpha M, c)$ is observable. It is left to show that $(A+\alpha M, \widehat{b})$ is controllable. Note that $P = A + \alpha N$ and $Q = A + \alpha M$ commute. This is because $Q = P + (2l\pi/\tau)J$, and $PJ = JP$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Using that $PQ = QP$ and therefore $QP^{-1} = P^{-1}Q$:

$$\mathcal{R}(Q, \widehat{b}) = (\widehat{b} \ Q\widehat{b}) = (QP^{-1}b \ QQP^{-1}b)$$
$$= (QP^{-1}b QP^{-1}Qb) = (QP^{-1})\mathcal{R}(Q, b)$$

Since M was chosen so that (Q, b) is controllable, it follows that $\mathcal{R}(Q, \widehat{b})$ is non-singular, and hence, $(A+\alpha M, \widehat{b})$ is controllable, as claimed. ∎

VII. CONCLUSION

For bilinear systems, we showed that step inputs are not enough for identification, nor do single pulses suffice, but that the family of all pulses (of a fixed amplitude but varying widths) do suffice. We presented results for single-input single-output systems, since one can obviously identify a multiple-input multiple-output system by considering each pair of input and output channels separately, and hence the family of pulses also works for the general case.

We emphasize that we dealt in this paper with ideal noise-free conditions, and ignored stochastic aspects and noisy data, because the underlying theoretical questions of what is ultimately achievable are easiest to understand in a deterministic setting. Tools such as those here have been used, however, in the formulation of identification algorithms from noisy data, for bilinear systems [24]. Nor did we deal here with questions of computational and sample complexity. However, the methods used are quite constructive and indeed have appeared in the same context in [12], where numerical implementations are studied; regarding sample complexity, we leave for further research the generalization of learning-theory results [25], [26] from the linear case to the classes of systems considered here.

Finally, bilinear systems were picked because an elegant result can be established for them, as well as their applicability and general interest. However, the study of similar problems to those treated here for more general classes of systems is of great interest.

APPENDIX

Proof of Lemma 4.1: Observe that for each canonical triple (A, b, c), the triple (A, c', b') is also canonical, and the two triples are i/o equivalent since $cA^k b = (cA^k b)' = b'(A')^k c'$ for all nonnegative integers k. Thus there is a (unique) similarity between (A, b, c) and (A, c', b'), an invertible matrix T such that:

$$AT = TA', \quad b = Tc', \quad cT = b'.$$

The formula for T is given in (11), which, since in this case $\widehat{A} = A'$ and $\widehat{b} = c'$ and $\mathcal{O}(A, c)' = \mathcal{R}(A', c')$, reduces to that shown. ∎

Although not needed, it is worth remarking that T is symmetric. This can be proved as follows: transposing the relations in (13), one has that also $AT' = T'A'$, $b = T'c'$, and $cT' = b'$. Since the similarity matrix T is unique, $T = T'$.

Proof of Lemma 4.2: The complement of $\mathcal{G}_0^c$ of $\mathcal{G}_0$ is the union of the solution sets of the following equations respectively:

$$\det \mathcal{R}(A, b) = 0$$
$$\det \mathcal{O}(A, c) = 0$$

and the n^2 scalar equations given by

$$N\tilde{T}(\sigma) = \tilde{T}(\sigma)N'. \tag{24}$$

Hence, $\mathcal{G}_0^c$ is an algebraic set. Each subset is proper (for the last one, pick an arbitrary canonical (A, b, c) and refer to the above remark that $\mathcal{B}(S)$ is always proper), and hence of dimension less than $n^2 + 2n$, so the union is also proper. ∎

Proof of Lemma 4.5: First note that $\mathcal{C} \subseteq \mathcal{G}_0$, and $\mathcal{C}^c$ is the union of $\mathcal{G}_0^c$ and of the solution sets of of the following equations:

$$\det \mathcal{R}(Q - N, b_0) = 0 \qquad (25)$$
$$\det \mathcal{O}(Q - N, c) = 0 \qquad (26)$$
$$\det(e^Q - I) = 0. \qquad (27)$$

Clearly, the solutions sets $\mathcal{A}_1$ and $\mathcal{A}_2$ of (25) and (26) respectively are proper algebraic sets. By the Spectral Mapping Theorem, (27) holds if and only if $-4k^2\pi^2$ is an eigenvalue of Q^2 for some integer k, and hence, the solution set $\mathcal{A}_3$ of (27) is the countable union of the solution sets $\{\mathcal{A}_{3k}\}$ of the equations

$$\det(Q^2 + 4k^2\pi^2 I) = 0, \qquad k = 0, 1, 2, \ldots.$$

Hence, $\mathcal{C}^c$ is the countable union of $\mathcal{G}_0^c$, $\mathcal{A}_1$, $\mathcal{A}_2$, and $\{\mathcal{A}_{3k}\}_{k \geq 0}$. ∎

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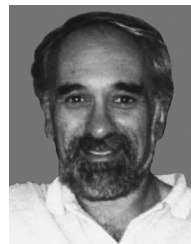
Eduardo D. Sontag (SM'87–F'93) received the Licenciado degree in mathematics from the University of Buenos Aires, Buenos Aires, Argentina, in 1972 and the Ph.D. degree in mathematics from the Center for Mathematical Systems Theory, University of Florida, Gainesville, in 1976.

Since 1977, he has been with the Department of Mathematics, Rutgers, The State University of New Jersey, Piscataway, where he is currently Professor II of Mathematics and a member of the Steering Committee, BioMapS Institute for Quantitative Biology. He is also a Member of the Graduate Faculties of the Department of Computer Science and of the Department of Electrical and Computer Engineering, and the Director of the Rutgers Center for Systems and Control. He has authored over 400 journal and conference papers and book chapters, as well as the three books. His major current research interests lie in several areas of systems molecular biology and control theory.

Dr. Sontag received the Reid Prize in Mathematics in 2001, the 2002 Hendrik W. Bode Lecture Prize from the IEEE, the 2002 Board of Trustees Award for Excellence in Research from Rutgers University, and the 2005 Teacher/Scholar Award from Rutgers University.

Yuan Wang (SM'07) received the Ph.D. degree in mathematics from Rutgers University, Piscataway, NJ, in 1990.

Since then she has been with the Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, where she is currently a Professor. Her research interests lie in several areas of control theory, including realization and stabilization of nonlinear systems.

Dr. Wang has served as an Associate Editor on the IEEE Conference Editorial Board and for *Systems & Control Letters*. Currently she serves as a Moderator of the optimization and control branch for the unified mathematics archive

Alexandre Megretski (M'94–SM'08) received the Ph.D. degree in control theory in from Leningrad University, Leningrad, Russia, in 1988.

He was a Researcher with the Royal Institute of Technology, Stockholm, Sweden, University of Newcastle, Australia, and a faculty member at Iowa State University, Ames. He is now a Professor of electrical engineering at MIT, Cambridge, MA, working on nonlinear systems (analysis, identification, model reduction, and design) and optimization.