



ELSEVIER

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

J. Differential Equations 224 (2006) 373–384

**Journal of
Differential
Equations**

www.elsevier.com/locate/jde

A cooperative system which does not satisfy the limit set dichotomy

Eduardo D. Sontag^{a,1}, Yuan Wang^{b,*,2}

^a*Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA*

^b*Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, FL 33431, USA*

Received 27 May 2005; revised 25 July 2005

Available online 22 September 2005

Abstract

The fundamental property of strongly monotone systems, and strongly cooperative systems in particular, is the limit set dichotomy due to Hirsch: if $x < y$, then either $\omega(x) < \omega(y)$, or $\omega(x) = \omega(y)$ and both sets consist of equilibria. We provide here a counterexample showing that this property need not hold for (non-strongly) cooperative systems.

© 2005 Elsevier Inc. All rights reserved.

Keywords: Monotone systems; Cooperative systems; Limit set dichotomy

1. Introduction

The field of cooperative, and more generally monotone systems, provides one of the most fruitful areas of theory as well as practical applications—particularly in biology—of dynamical systems. For an excellent introduction, see the textbook by Smith [4] and the recent exposition [3]. One of its central tools is a classical theorem of Hirsch [1,2], the “limit set dichotomy” for strongly monotone (in particular, strongly cooperative)

* Corresponding author. Fax: +1 561 297 2436.

E-mail addresses: sontag@math.rutgers.edu (E.D. Sontag), ywang@math.fau.edu (Y. Wang).

¹ Partially supported by NSF Grants EIA-0205116 and DMS-0504557.

² Partially supported by NSF Grants DMS-0072620 and DMS-0504296 and Chinese National Natural Science Foundation Grant 60228003.

systems, see Theorem 1.16 in [3]. The limit set dichotomy states that if $x < y$, then either $\omega(x) < \omega(y)$, or $\omega(x) = \omega(y)$ and both sets consist of equilibria.

According to the recent survey [3], the problem of deciding if there are any cooperative systems for which the dichotomy fails is still open. In [3], Example 1.24, one finds a system which is monotone but not strongly monotone, for which the dichotomy fails. The order in this example is the “ice cream cone” order, and the authors explicitly state that it is unknown whether a polyhedral cone example exists. A cooperative system is one defined by a set of ordinary differential equations $\dot{x} = f(x)$, where $f = (f_1, \dots, f_n)'$, with the property that $\frac{\partial f_i}{\partial x_j}(x) \geq 0$ for all $i \neq j$ and all x . Cooperative systems are monotone with respect to a polyhedral cone, namely the main orthant in \mathbb{R}^n . Thus, a counterexample using cooperative systems provides an answer to this open question. We provide such a counterexample here.

To be precise, we construct here two differentiable functions

$$f, g : \mathbb{R} \rightarrow \mathbb{R}$$

such that $f(0) = g(0) = 0$, $xf(x) < 0$ and $yg(y) < 0$ for all $x, y \neq 0$, and consider essentially the following system:

$$\begin{aligned}\dot{x} &= f(x), \\ \dot{y} &= g(y), \\ \dot{z} &= x + y.\end{aligned}$$

This system is cooperative. Note that solutions of the x and y equations converge to zero as $t \rightarrow \infty$. Moreover, for this system, and for any $\delta > 0$, the following property holds:

There is a solution X with initial condition $(x(0), y(0), z(0))'$ such that $|x(0)| < \delta$ and $|y(0)| < \delta$ so that the omega-limit set $\omega(X)$ is compact and it contains the set

$$\{(0, 0, \zeta) \mid z(0) - \frac{1}{2} \leq \zeta \leq z(0) + \frac{1}{2}\}.$$

The limit set dichotomy states would imply that, for any initial conditions

$$(x(0), y(0), z(0))' \quad \text{and} \quad (\hat{x}(0), \hat{y}(0), \hat{z}(0))'$$

for which $x(0) \leq \hat{x}(0)$, $y(0) \leq \hat{y}(0)$, and $z(0) \leq \hat{z}(0)$, and with at least one of the inequalities being strict, the corresponding solutions X, \hat{X} have the property that either $\omega(X) = \omega(\hat{X})$ or $\omega(X) < \omega(\hat{X})$. This last property implies in particular that $\omega(X)$ and $\omega(\hat{X})$ are disjoint. Now, for our example, clearly $\omega(X) \neq \omega(\hat{X})$ as long as $z(0) \neq \hat{z}(0)$ (since the z -components of solutions are translates by $z(0)$ of the solutions with $z(0) = 0$), but the omega-sets intersect as long as the x and y initial conditions are as discussed above, $\hat{z}(0) \leq z(0)$, and

$$z(0) - \frac{1}{2} < \hat{z}(0) + \frac{1}{2},$$

i.e.

$$0 \leq z(0) - \hat{z}(0) \leq 1.$$

Thus we contradict the limit set dichotomy.

Let us comment briefly on the significance of our counterexample, and on a problem still left open. Our example shows that the limit set dichotomy does not generally hold for cooperative systems, i.e. monotone systems with respect to the standard positive orthant, providing a counterexample to the limit set dichotomy question, and also to the nonordering of limit sets [3]. On the other hand, the limit set dichotomy is one of the key tools in showing that the generic orbit of a strongly cooperative system (or more generally, strongly order preserving semiflow) is *quasiconvergent*, i.e., its omega limit set consists only of equilibria. For non-strongly cooperative systems, it is still an open question whether generic quasiconvergence holds, because, in our example, every solution is quasiconvergent. In other words, although the limit set dichotomy fails, generic quasiconvergence, which is one of the reasons for desiring the limit set dichotomy in the first place, does indeed hold. (In addition, our system has a degeneracy in that its equilibrium set, the z -axis, is not normally hyperbolic since it consists of points whose Jacobian matrix has three zero eigenvalues.) On the other hand, our example provides a continuously differentiable vector field, and for strongly monotone such vector fields (whose linearized systems are also strongly monotone), there is also a result on generic convergence to equilibria, not merely quasiconvergence, see [4, Chapter 2, Theorem 4.7]. The question of finding a cooperative system for which the generic quasiconvergence theorem holds, is still open, but we have provided cooperative counterexamples to the dichotomy and nonordering results, as well as (see Remark 2.6) to Theorem 4.7 in [4], for non-strongly monotone systems.

2. The example

To present an example as discussed before, we first consider the following result:

Lemma 2.1. *For any $\delta > 0$, there exist two C^1 functions $p, q : [-1, \infty) \rightarrow [0, \infty)$ such that the following hold:*

- (1) *both p and q are strictly decreasing functions;*
- (2) *$p(0) < \delta, q(0) < \delta$;*
- (3) *$p(t) \rightarrow 0$ and $q(t) \rightarrow 0$;*
- (4) *for any $a, b \in (-1, 1)$, the function $H_{a,b}$ defined by*

$$H_{a,b}(T) = \int_0^T (p(t+a) - q(t+b)) dt$$

is bounded; and

- (5) *for any $a, b \in (-1, 1)$,*

$$\overline{\lim}_{T \rightarrow \infty} H_{a,b}(T) > 0, \quad \underline{\lim}_{T \rightarrow \infty} H_{a,b}(T) < 0;$$

and

$$\overline{\lim}_{T \rightarrow \infty} H_{a,b}(T) - \underline{\lim}_{T \rightarrow \infty} H_{a,b}(T) \geq 1.$$

We will prove the lemma in Section 3.1 by showing that the two functions can be chosen as

$$p(t) = \frac{1}{\sqrt{t+c_0}}, \quad (1)$$

$$q(t) = \frac{1}{\sqrt{t+c_0}} + \frac{\sin[(t+c_0)^{1/4}]}{(t+c_0)^{3/4}}, \quad (2)$$

where c_0 can be any number in $[\alpha, \infty)$ for some α to be chosen later.

Below we show that, for some C^1 maps f and g , for $a \in (-1, 1)$, $p(t+a)$ is an solution of $\dot{x} = f(x)$; $q(t+a)$ and $-q(t+a)$ are solutions of $\dot{y} = g(y)$.

It can be seen readily that, for $|a| < 1$, $p(t+a)$ is the solution of the initial value problem

$$\dot{x} = -\frac{x^3}{2}, \quad x(0) = \frac{1}{\sqrt{c_0+a}}.$$

Let $\psi : (-1, \infty) \rightarrow (-\infty, 0)$ be defined by $\psi(t) = \frac{d}{dt}q(t)$, and $\varphi : (0, \rho) \rightarrow (-1, \infty)$ be defined by $\varphi(r) = q^{-1}(r)$, where $\rho = q(-1) \geq q(t)$ for all $t > -1$. Let $g(r) = \psi \circ \varphi(r)$ for $r \neq 0$, and $g(0) = 0$.

Lemma 2.2. *The function $g : [0, \rho) \rightarrow (-\infty, 0)$ is of class C^1 .*

Extend g from $[0, \rho)$ to $[0, \infty)$ as a C^1 function, and then extend g to \mathbb{R} by letting $g(-r) = -g(r)$ for $r < 0$. Still denote the newly extended function by g . Then g is a C^1 function. Let $f(x) = -x^3/2$.

Lemma 2.3. *For any $a, b \in (-1, 1)$, $(x_a(t), y_b(t))'$ defined by*

$$x_a(t) = p(t+a), \quad y_b(t) = -q(t+b)$$

for $a, b \in (-1, 1)$ is the solution to the initial value problem

$$\begin{aligned} \dot{x} &= f(x), & x_a(0) &= \frac{1}{\sqrt{c_0+a}}, \\ \dot{y} &= g(y), & y_b(0) &= -q(c_0+b). \end{aligned} \quad (3)$$

To get a system as discussed in Section 1, we would like to cascade system (3) with the one-dimensional system $\dot{z} = x + y$. To obtain a system for which all trajectories are bounded, we choose a C^1 function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ with the property such that

- $\sigma(r) = 0$ for all $|r| \leq 1 + M$, where $M = \sup\{|H_{a,b}(t)| : |a| \leq 1, |b| \leq 1, t \geq 0\}$;
- $r\sigma(r) > 0$ for all $|r| > M + 1$; and
- σ is proper.

Consider the system

$$\begin{aligned} \dot{x} &= f(x), \\ \dot{y} &= g(y), \\ \dot{z} &= x + y - \sigma(z). \end{aligned} \tag{4}$$

It is clear that system (4) is cooperative.

Lemma 2.4. Consider system (4):

- (1) the (x, y) subsystem is globally asymptotically stable; and
- (2) every trajectory of system (4) has a compact closure.

Below we present our final result. We use $X(t) = (x(t), y(t), z(t))^T$ to denote a solution of system (4).

Proposition 2.5. Consider the cooperative system (4). For any $\delta > 0$, there exist two trajectories $X_1(t)$ and $X_2(t)$ with $X_1(0) < X_2(0)$ and $|X_1(0)| < \delta, |X_2(0)| < \delta$ such that $\omega(X_1) \neq \omega(X_2)$ and $\omega(X_1) \leq \omega(X_2)$ fails.

Remark 2.6. In fact, we have obtained a system for which the statement of Proposition 2.5 can be made generic in the following sense. For any given $\delta_0 > 0$, there exists $|X_0| < \delta$ and some $\delta_1 > 0$ such that for any pair of trajectories X_1, X_2 of system (4) satisfying $|X_1(0) - X_0| < \delta_1, |X_2(0) - X_0| < \delta_1$, and $z_1(0) \neq z_2(0)$, it holds that $\omega(X_1) \neq \omega(X_2)$ and $\omega(X_1) \leq \omega(X_2)$ fails.

Proof of Proposition 2.5. Assume that $\delta > 0$ is given. Choose c_0 as in (1)–(2) large enough so that $\frac{1}{\sqrt{c_0-1}} < \delta$ and $q(c_0 - 1) < \delta$. Then, for any trajectory of the system

$$\dot{x} = f(x), \quad \dot{y} = g(y)$$

with

$$\frac{1}{\sqrt{c_0 + 1}} < x(0) < \frac{1}{\sqrt{c_0 - 1}}, \quad -q(-1) < y(0) < -q(1), \tag{5}$$

one has $x(t) = p(t + a)$, $y(t) = -q(t + b)$ for some $a, b \in (-1, 1)$, and hence,

$$\begin{aligned} & \overline{\lim}_{t \rightarrow \infty} \int_0^t (x(s) + y(s)) ds - \underline{\lim}_{t \rightarrow \infty} \int_0^t (x(s) + y(s)) ds \\ &= \overline{\lim}_{t \rightarrow \infty} H_{a,b}(t) - \underline{\lim}_{t \rightarrow \infty} H_{a,b}(t) \geq 1. \end{aligned} \tag{6}$$

Let $x_0 = 1/\sqrt{c_0}$, $y_0 = q(0)$, and

$$\delta_1 = \min \left\{ \frac{1}{\sqrt{c_0 - 1}} - \frac{1}{\sqrt{c_0}}, \frac{1}{\sqrt{c_0}} - \frac{1}{\sqrt{c_0 + 1}}, q(-1) - q(0), q(0) - 1(1) \right\}.$$

Then (6) holds for any trajectory $(x(t), y(t))^T$ of (5) with $|x(0) - x_0| < \delta_1$, $|y(0) - y_0| < \delta_1$. Take any $|z_0| < 1$,

$$\left| z_0 + \int_0^t (x(s) + y(s)) ds \right| = |z_0 + H_{a,b}(t)| < M + 1,$$

and therefore, $z_0 + \int_0^t (x(s) + y(s)) ds$ is the solution of the z -subsystem of (4) with the initial value $z(0) = z_0$ (note that $\sigma(s) = 0$ when $|s| \leq M + 1$). It then follows from statement (v) of Lemma 2.1 that for any $x(0), y(0)$ satisfying (5), it holds that

$$\overline{\lim}_{t \rightarrow \infty} (z(t) - z(0)) > 0, \quad \underline{\lim}_{t \rightarrow \infty} (z(t) - z(0)) < 0,$$

and

$$\left[\overline{\lim}_{t \rightarrow \infty} (z(t) - z(0)) \right] - \left[\underline{\lim}_{t \rightarrow \infty} (z(t) - z(0)) \right] > 1.$$

Observe that for any trajectory $X(t)$ of system (4),

$$\omega(X) \supseteq \left\{ (0, 0, \alpha) : \underline{\lim}_{t \rightarrow \infty} z(t) < \alpha < \overline{\lim}_{t \rightarrow \infty} z(t) \right\}.$$

For any two solutions $X_1(t) := (x(t), y(t), \hat{z}(t))$ and $X_2(t) := (x(t), y(t), \hat{z}(t))$, where $x(0)$ and $y(0)$ satisfy (5), and $\hat{z}(0) \neq z(0)$, the sets of ω -limit points of $\{z(t) : t \geq 0\}$ and $\{\hat{z}(t) : t \geq 0\}$ are different (since $z(t) - \hat{z}(t) \equiv z(0) - \hat{z}(0)$). Moreover, with $\hat{z}(0) < z(0)$,

$$\begin{aligned} \overline{\lim}_{t \rightarrow \infty} \hat{z}(t) - \underline{\lim}_{t \rightarrow \infty} z(t) &= \left[\overline{\lim}_{t \rightarrow \infty} (\hat{z}(t) - \hat{z}(0)) \right] - \left[\underline{\lim}_{t \rightarrow \infty} (z(t) - z(0)) \right] \\ &\quad - (z(0) - \hat{z}(0)) \geq 1 - (z(0) - \hat{z}(0)) > 0 \end{aligned}$$

if $0 < z(0) - \hat{z}(0) < 1$. Hence, $\omega(\hat{X}) \leq \omega(X)$ fails. \square

3. Proofs of the lemmas

In this section, we provide proofs of the results.

3.1. Proof of Lemma 2.1

First we let

$$p_0(t) = \frac{1}{\sqrt{t}}, \quad q_0(t) = \frac{1}{\sqrt{t}} + \frac{1}{t^{3/4}} \sin t^{1/4} \quad t \geq 1. \tag{7}$$

To show that q_0 is decreasing, we consider $q'_0(t)$:

$$\begin{aligned} q'_0(t) &= -\frac{1}{2t^{3/2}} - \frac{3}{4t^{7/4}} \sin t^{1/4} + \frac{1}{t^{3/4}} \cdot \frac{1}{4t^{3/4}} \cos t^{1/4} \\ &= -\frac{1}{2t^{3/2}} - \frac{3}{4t^{7/4}} \sin t^{1/4} + \frac{1}{4t^{3/2}} \cos t^{1/4} \\ &\leq -\frac{1}{4t^{3/2}} + \frac{3}{4t^{7/4}}. \end{aligned} \tag{8}$$

So, $q'_0(t) \leq 0$ when $\frac{1}{t^{3/2}} - \frac{3}{t^{7/4}} \geq 0$. This is the same as $t^{3/2} \leq t^{7/4}/3$, or $t^{1/4} \geq 3$, that is, $t \geq 81$.

Let $p(t) = p_0(t + c_0)$, $q(t) = q_0(t + c_0)$, where $c_0 \geq 81$ will be chosen later on. Now both p and q are differentiable on $[0, \infty)$ and monotonically decrease to 0. For $a \in (-1, 1)$ and $b \in (-1, 1)$, let $H_{a,b}(T)$ be as defined as in Lemma 2.1. Then

$$\begin{aligned} H_{a,b}(T) &= \int_0^T (p(t+a) - q(t+b)) dt \\ &= \int_0^T \frac{1}{\sqrt{t+c_0+a}} - \frac{1}{\sqrt{t+c_0+b}} - \frac{1}{(t+c_0+b)^{3/4}} \sin(t+c_0+b)^{1/4} dt \\ &= \int_0^T \frac{\sqrt{t+c_0+b} - \sqrt{t+c_0+a}}{\sqrt{t+c_0+a}\sqrt{t+c_0+b}} - \frac{1}{(t+c_0+b)^{3/4}} \sin(t+c_0+b)^{1/4} dt. \end{aligned} \tag{9}$$

Now, for the first term in (9), we have

$$\begin{aligned} &\left| \frac{\sqrt{t+c_0+b} - \sqrt{t+c_0+a}}{\sqrt{t+c_0+a}\sqrt{t+c_0+b}} \right| \\ &= \frac{b-a}{\sqrt{t+c_0+a}\sqrt{t+c_0+b}(\sqrt{t+c_0+b} + \sqrt{t+c_0+a})} \\ &\leq \frac{|b-a|}{2(t+c_0-1)^{3/2}} \quad \forall |a| < 1, \quad |b| < 1, \end{aligned}$$

and hence, the integral

$$\int_0^\infty \frac{\sqrt{t + c_0 + b} - \sqrt{t + c_0 + a}}{\sqrt{t + c_0 + a}\sqrt{t + c_0 + b}} dt$$

is convergent, and for $|a| < 1, |b| < 1,$

$$\left| \int_0^T \frac{\sqrt{t + c_0 + b} - \sqrt{t + c_0 + a}}{\sqrt{t + c_0 + a}\sqrt{t + c_0 + b}} dt \right| \leq \frac{2}{\sqrt{c_0}} \quad \forall T > 0. \tag{10}$$

For the second term in (9), using $u = (t + c_0 + b)^{1/4},$ one has

$$\begin{aligned} \int_0^T \frac{1}{(t + c_0 + b)^{3/4}} \sin(t + c_0 + b)^{1/4} dt &= \int_{(c_0+b)^{1/4}}^{(T+c_0+b)^{1/4}} \sin u du \\ &= -\cos(T + c_0 + b)^{1/4} + \cos(c_0 + b)^{1/4}. \end{aligned}$$

Combining this with (10), one sees that, for any $|a| < 1, |b| < 1, H_{a,b}(t)$ is bounded on $[0, \infty).$

Let $c_0 \geq 82$ be of the form $(2k\pi + \pi/2)^4.$ For $|b| < 1,$

$$\frac{d}{db}(c_0 + b)^{1/4} = \frac{1}{4(c_0 + b)^{3/4}} \leq \frac{1}{4} \leq \frac{\pi}{6}.$$

Consequently,

$$c_0^{1/4} - \frac{\pi}{6} \leq (c_0 + b)^{1/4} \leq c_0^{1/4} + \frac{\pi}{6} \quad \forall |b| < 1.$$

This implies that

$$-\frac{1}{2} \leq \cos(c_0 + b)^{1/4} \leq \frac{1}{2} \quad \forall |b| < 1.$$

Thus,

$$\begin{aligned} \overline{\lim}_{T \rightarrow \infty} \int_0^T \frac{1}{(t + c_0 + b)^{3/4}} \sin(t + c_0 + b)^{1/4} dt &= 1 - \cos(c_0 + b)^{1/4} \geq \frac{1}{2}, \\ \underline{\lim}_{T \rightarrow \infty} \int_0^T \frac{1}{(t + c_0 + b)^{3/4}} \sin(t + c_0 + b)^{1/4} dt &= -1 - \cos(c_0 + b)^{1/4} \leq -\frac{1}{2}. \end{aligned}$$

Finally, we let $c_0 = (2k\pi + \pi/2)^4$ with k large enough so that

- $c_0 \geq 82$; and
- $\frac{2}{\sqrt{c_0}} < 1/4$.

This way, we get for all $|a| < 1, |b| < 1$,

$$\begin{aligned} \overline{\lim}_{T \rightarrow \infty} H_{a,b}(T) &= \lim_{T \rightarrow \infty} \int_0^T \left(\frac{1}{\sqrt{t+c_0+a}} - \frac{1}{\sqrt{t+c_0+b}} \right) dt \\ &\quad - \underline{\lim}_{T \rightarrow \infty} \int_0^T \frac{1}{(t+c_0+b)^{3/4}} \sin(t+c_0+b)^{1/4} dt \\ &\geq l_{a,b} + \frac{1}{2}, \end{aligned}$$

where $l_{a,b} = \int_0^\infty \left(\frac{1}{\sqrt{t+c_0+a}} - \frac{1}{\sqrt{t+c_0+b}} \right) dt$; and

$$\begin{aligned} \underline{\lim}_{T \rightarrow \infty} H_{a,b}(T) &= \lim_{T \rightarrow \infty} \int_0^T \left(\frac{1}{\sqrt{t+c_0+a}} - \frac{1}{\sqrt{t+c_0+b}} \right) dt \\ &\quad - \overline{\lim}_{T \rightarrow \infty} \int_0^T \frac{1}{(t+c_0+b)^{3/4}} \sin(t+c_0+b)^{1/4} dt \\ &\leq l_{a,b} - \frac{1}{2}, \end{aligned}$$

which implies that

$$\overline{\lim}_{t \rightarrow \infty} H_{a,b}(t) - \underline{\lim}_{t \rightarrow \infty} H_{a,b}(t) \geq 1.$$

Since c_0 was chosen so that $|l_{a,b}| < 1/4$, one has

$$\underline{\lim}_{T \rightarrow \infty} H_{a,b}(T) < -1/4, \quad \overline{\lim}_{T \rightarrow \infty} H_{a,b}(T) > 1/4.$$

3.2. Proof of Lemma 2.2

First of all, it can be calculated (see also (8)) that, for t large enough

$$|\psi(t)| = \left| \frac{d}{dt} q_0(t+c_0) \right| \leq \frac{M}{t^{3/2}} \tag{11}$$

for some $M \geq 0$, where q_0 is defined as in (7). Let $\varphi : (0, \rho) \rightarrow (-1, \infty)$ be defined by $\varphi(r) = q^{-1}(r)$, where $\rho = q(-1) \geq q(t)$ for all $t > -1$. Note that

$$\lim_{r \rightarrow 0} \varphi(r) = \infty.$$

Let $g(r) = \psi \circ \varphi(r)$ for $r \neq 0$, and $g(0) = 0$. Then g is continuous on $[0, \rho)$, and of C^1 on $(0, \rho)$. Observe that $g(r) < 0$ for all $r \in (0, \rho)$. Below we show that g is differentiable at 0.

Fact 1. *There exist some $\delta_0 > 0$ and some $L_0 > 0$ such that*

$$\varphi(r) \geq \frac{L_0}{r^2} \quad \forall r \in (0, \delta_0). \quad (12)$$

To prove Fact 1, write $t = q^{-1}(r)$. Then $r = q(t)$, and

$$r \geq \frac{1}{\sqrt{t+c_0}} - \frac{1}{(t+c_0)^{3/4}} \geq \frac{L_0}{\sqrt{t}} \quad \forall t \geq T_0 \quad (13)$$

for some $L_0 > 0$ and some $T_0 \geq 0$. Since $q(t)$ decreases to 0 as $t \rightarrow \infty$, it follows that for some $\delta_0 > 0$, it holds that $t \geq \frac{L_0}{r^2}$ for all $r \in (0, \delta_0)$, this is, (12) holds.

Fact 2. $g'(0) = 0$.

Fact 2 follows from Fact 1 combined with (11):

$$|\psi(\varphi(r))| \leq \frac{M}{[\varphi(r)]^{3/2}} \leq M \left[\frac{L_0}{r^2} \right]^{-3/2} \leq \tilde{M} r^3$$

for all $r > 0$ in a neighborhood of 0, where $\tilde{M} > 0$ is some constant. This shows that g is differentiable at 0, and $g'(0) = 0$.

Fact 3. $q'(r)$ is continuous at $r = 0$.

To prove this fact, we first get an estimate on $\psi'(t)$ for t large enough.

$$\begin{aligned} \psi'(t-c_0) = q_0''(t) &= \frac{3}{4t^{5/2}} + \frac{21}{16t^{11/4}} \sin t^{1/4} \\ &\quad - \frac{3}{16t^{10/4}} \cos t^{1/4} - \frac{3}{8t^{5/2}} \cos t^{1/4} - \frac{1}{16t^{9/4}} \sin t^{1/4}. \end{aligned}$$

Hence, for some $T_1 > 0$ and some $L_1 \geq 0$,

$$\left| \frac{d}{dt} \psi(t) \right| \leq \frac{L_1}{t^{9/4}} \quad \forall t \geq T_1.$$

This implies that, for some $\delta_1 > 0$,

$$|\psi'(\varphi(r))| \leq \frac{L_1}{[\varphi(r)]^{9/4}} \quad \forall r \in (0, \delta_1). \tag{14}$$

We also need the following estimate on $\varphi'(r)$ near 0:

$$\begin{aligned} \left| \frac{d}{dt} q(t - c_0) \right| &= \left| \frac{d}{dt} q_0(t) \right| \geq \frac{1}{2t^{3/2}} - \frac{3}{4t^{7/4}} - \frac{1}{4t^{3/2}} \\ &= \frac{1}{4t^{3/2}} - \frac{3}{4t^{7/4}} \quad \forall t \geq c_0. \end{aligned}$$

It then can be seen that for some $T_2 > 0$ and some $L_2 > 0$, one has

$$|q'(t)| \geq \frac{L_2}{t^{3/2}} \quad \forall t \geq T_2,$$

which implies that for some $\delta_2 > 0$,

$$|q'(\varphi(r))| \geq \frac{L_2}{[\varphi(r)]^{3/2}} \quad \forall r \in (0, \delta_2).$$

Finally,

$$\begin{aligned} |g'(r)| &= |\psi'(\varphi(r))\varphi'(r)| = \left| \psi'(\varphi(r)) \frac{1}{G'(\varphi(r))} \right| \\ &\leq \frac{L_1}{[\varphi(r)]^{9/4}} \cdot \frac{1}{\frac{L_2}{[\varphi(r)]^{3/2}}} = \frac{L_1}{L_2[\varphi(r)]^{3/4}} \rightarrow 0 \quad \text{as } r \rightarrow 0. \end{aligned}$$

Hence, $g'(r)$ is continuous at $r = 0$. With this we conclude that g is of C^1 on $[0, \rho)$.

3.3. Proof of Lemma 2.3

The statement about $x_a(t)$ is certainly clear.

To treat the part about $y_b(t)$, first observe that $q'(t)$ can be written as $q'(\varphi(q(t)))$. Also note that for any $|b| < 1$, $0 \leq q(t+b) \leq q(-1)$ for all $t \geq 0$, that is, $q(t+b) \in (0, \rho)$ for all $t \geq 0$. Hence, we have

$$\frac{d}{dt} q(t) = g(q(t)),$$

that is, $q(t)$ is a solution of the differential equation $\dot{y} = q(y)$. Let $\tilde{q}(t) = -q(t)$. Then

$$\frac{d\tilde{q}(t)}{dt} = -\frac{dq}{dt} = -g(q(t)) = -g(-\tilde{q}(t)) = g(\tilde{q}(t)).$$

This shows that $-q(t)$ is also a solution of the equation $\dot{y} = g(y)$.

3.4. Proof of Lemma 2.4

Since for $xf(x) < 0$ and $yg(y) < 0$ for all $x \neq 0, y \neq 0$, both the x - and the y -subsystems are globally asymptotically stable.

To complete the proof, it is enough to note that every trajectory of the system

$$\dot{z} = -\sigma(z) + h(t)$$

is bounded for any choice of bounded function $h(t)$.

References

- [1] M.W. Hirsch, Convergence in ordinary and partial differential equations, Notes for Colloquium Lectures at University of Toronto, August 23–26, 1982, American Mathematical Society, Providence, RI, 1982.
- [2] M.W. Hirsch, Differential equations and convergence almost everywhere in strongly monotone semiflows, *Contemp. Math.* 17 (1983) 267–285.
- [3] M.W. Hirsch, H.L. Smith, Monotone dynamical systems, in: *Handbook of Differential Equations, Ordinary Differential Equations*, vol. 2, Elsevier, Amsterdam, 2005.
- [4] H.L. Smith, *Monotone Dynamical Systems, an Introduction to the Theory of Competitive and Cooperative Systems*, *Mathematical Surveys and Monographs*, vol. 41, American Mathematical Society, Providence, RI, 1995.