On Generalized Inverses of Polynomial and Other Matrices

EDUARDO D. SONTAG

Abstract—**Necessary and sufficient conditions are given for a rectangular multivariable polynomial matrix to have a (weak) generalized inverse, extending recent results for single-variable polynomial matrices obtained by other authors.**

I. Introduction

The notion of the generalized inverse of a real or complex matrix has found many applications in various areas of system theory. Motivated by the emerging theory of multidimensional systems [3] and corresponding problems for networks and large-scale systems, Bose and Mitra [4] discussed in a recent article the extension of the idea of (weak) generalized inverse to the case of matrices whose entries belong to a polynomial ring or to the ring of integers.

Using reduction to Smith form, Bose and Mitra characterized those matrices admitting a weak generalized inverse in the cases of the rings of integers and of polynomials in a single variable. For the general case of interest, that of polynomials in several variables, they proved only partial results and left the solution open. (Independently of that work, Batigne [1] obtained the same results for the case of integer matrices, with analogous methods.)

We give here a complete characterization for the case of polynomials in several variables. The gist of the solution is that a generalized inverse *forces* the existence of a "Smith form" for the original matrix. The paper will treat the even more general case of matrices (and other linear transformations) over arbitrary integral domains, both because this is the natural way of posing the corresponding questions and because of the various rings that appear in the theory of linear systems over rings [13], [9]; the treatment with the added level of generality has the same degree of difficulty as that for polynomials.

In the case of rings of rational functions with no real poles, we shall also construct a (full) generalized inverse, and will sketch an application to an optimization problem arising in the study of families of systems.

II. Definitions and Statement of Results

Throughout this paper, R will denote an arbitrary integral domain. Of particular interest will be the example $R=\mathbb{C}[z_1,\cdots,z_n]$, the ring of polynomials in n variables with complex coefficients. A ring of real rational functions will appear later.

A *weak generalized inverse* (WGI) of a (nonnecessarily square) matrix A (also called a "{1,2}-generalized inverse" of A) is any matrix B such that the following equations hold:

$$ABA=A \tag{1}$$

$$BAB=B \tag{2}$$

(cf. [2]). If A and B_0 satisfy (1), then $B:=B_0AB_0$ is WGI of A. Thus, A has a WGI iff there is some B satisfying (1).

For polynomial rings the main result will be the following.

Theorem 1: The following statements are equivalent for a matrix $A=A(z_1,\cdots,z_n)$ *over* $R=\mathbb{C}[z_1,\cdots,z_n]$.

a) *A has a WGI.*

b) *There exist square unimodular (= nonzero scalar determinant) matrices P,Q over R such that $A=PA_0Q$, with*

$$A_0=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r is the identity matrix of order $r=\text{rank } A$.

Manuscript received June 18, 1979; revised December 10, 1979. Paper recommended by B. Francis, Chairman of the Linear Systems Committee. This work was supported in part by the U.S. Air Force under Grant AFOSR-F49620-79-C-0117.

The author is with the Department of Mathematics, Rutgers University, New Brunswick, NJ 08903.

0018-9286/80/0600-0514$00.75

c) *As a function of the complex variables* $(z_1,\cdots,z_n)$, *the rank of* $A(z_1,\cdots,z_n)$ *is constant.*

This result will follow from a more general result below, which applies to general domains R. In particular, $R=\mathbb{Z}[z_1,\cdots,z_n]$ [4], the statement in part b) above will remain the same except for the fact that "unimodular" will now mean determinant $=\pm 1$. For the case of $R=\mathbb{R}[z_1,\cdots,z_n]$, the result will remain the same as Theorem 1 except that P,Q in part b) will be matrices of *real* polynomials. Note that it is immediate that part b) implies a), since

$$B:Q^{-1}A_0P^{-1}$$

is then a WGI of A.

The sequel will involve concepts from commutative algebra, a good reference for which is [5]. All undefined terms will be found in this reference. "Module" will always mean R-module, "linear" will be R-linear. An *R-field* will be any R-algebra which is a field. The tensor product "$\otimes$" will mean tensor product as R-modules or of R-linear maps, and "projective" will mean finitely generated projective R-module. Composition of maps will be denoted by juxtaposition. Throughout, M and N will be two arbitrary but fixed projective modules and $A:M\to N$ a linear map; $A\otimes K$ will be the map $A\otimes 1_K$. As before, a WGI of A will be any linear $B:N\to M$ satisfying (1), (2).

Definition: A *splits* iff the image of A is a direct summand of N.

Remark: The above means that A splits iff there is a factorization

$$A=DC \tag{3}$$

with

$$C:M\to S \text{ onto} \tag{4}$$

and

$$D:S\to N \text{ one-to-one} \tag{5}$$

such that, further, S is projective and there exist $C_1:S\to M$ and $D_1:N\to S$ with $CC_1=D_1D=$ identity of S. Equivalently, A splits iff

$$\text{coker } A=N/\text{Im } A \text{ is projective.} \tag{6}$$

The main result is then the following.

Theorem 2: The following statements are equivalent.

a) *A has a WGI.*

b) *A splits.*

c) $A\otimes K$ has a constant rank, independent of the particular R-field K.

The statement c) can be simplified somewhat in the (present) case of $R=$ integral domain, although the general statement is desirable since the proof will be valid in more generality (R with connected spectrum). Specifically, it is enough to consider the cases $K=$ quotient field of R and $K=$ all fields of the type R/J, J a maximal ideal of R. Furthermore, if R is Jacobson-semisimple (= the intersection of all maximal ideals is zero), it is enough to check c) for the fields R/J. Thus, for $R=\mathbb{C}[z_1,\cdots,z_n]$, which is semisimple, c) corresponds to checking the rank of $A\otimes K$ for each $J=$ ideal of functions zero at some point of $\mathbb{C}^n$, by Hilbert's Nullstellensatz. When A is a matrix, this becomes condition c) of Theorem 1.

Conditions b) of Theorems 1 and 2 are also equivalent for polynomial rings. Indeed, by the Quillen–Suslin theorem [10], the S in (4)–(5) is free, and bases can be chosen for M,N so that A has the matrix A_0 given in (6) of Theorem 1; P,Q are the corresponding base change matrices. *Thus, Theorem 1 follows from Theorem 2.*

III. Proof of Theorem 2

We shall prove first that b) is equivalent to c).

b) implies c): With the notations in (4)–(5), the fact that R is a domain (having, therefore, a connected spectrum and thus the rank being well-defined for projectives) implies together with b) that the K-vector space $S\otimes K$ has constant dimension. Further, the fact that A splits implies that $C\otimes K$ is onto and $D\otimes K$ is one-to-one. The dimension of $S\otimes K$ is then equal to the rank of A, since $A\otimes K$ has the full rank factorization

$$A\otimes K=(D\otimes K)(C\otimes K)$$

by functoriality of $-\otimes K$. Therefore, the rank of $A\otimes K$ is also independent of K.

c) implies b): Using the characterization in (6), it must be proved that coker A is projective. Projectives being locally free [5, Theorem II.5.2.1]), one may consider the various local rings R_j obtained localizing R, and prove that (coker $A)\otimes R_j$ is free. But $-\otimes R_j$ being flat, the latter is the cokernel of $A\otimes R_j$. The hypothesis c) is again true over R_j (an R_j-field is also an R-field), so

$$(\text{coker } A\otimes R_j)\otimes K=\text{coker } A\otimes K$$

has constant dimension. This is the situation in [5, Proposition II.3.2.7], form which one concludes that (coker $A)\otimes R_j$ is free, as wanted.

We now prove that a) and b) are equivalent.

b) implies a): With the notations in (4)–(5), let $B:C_1D_1$. Then

$$ABA=(DC)(C_1D_1)(DC)=D(CC_1)(D_1D)C=DC=A$$

and

$$BAB=(C_1D)(DC)(C_1D_1)=C_1(D_1D)(CC_1)D_1=C_1D_1=B.$$

Thus, B is WGI of A.

a) implies b): Let B be such that $ABA=A$. Denote

$$L:=AB:N\to N.$$

Then,

$$LL=(AB)(AB)=(ABA)B=AB=L,$$

so L is idempotent. Therefore,

$$S:=\text{image of } L$$

is a direct summand of N. But the equalities $L=AB$ and $A=LA$ imply that A and L have the same image; so the image of A is a factor of N as wanted.

Remark: 1) There are various ways of checking the criterion c) when A is a matrix ($M,N=$ free). For example, if A is of rank r, then c) will hold if and only if the ideal generated by all the r minors of A is the unit ideal of R, a condition that can be checked for polynomial rings via resultants. When R is a principal-ideal domain (PID) this condition is equivalent to requiring that the greatest common divisor of all the r-minors be a unit. In the PID case, moreover, the conditions are equivalent to all the invariant factors of A being units; this is in fact the characterization obtained by Bose and Mitra and by Batigne for the integers and for polynomials in a single variable.

2) For the case of $R=\mathbb{C}[z_1,\cdots,z_n]$, the condition given for the existence of a WGI is one of no common zeros of the r-minors, which define hypersurfaces in $\mathbb{C}^n$. Whether having a WGI is a "generic" property or not (for arbitrary A of a given size) depends therefore on the relation between the dimension of A and the number of variables n.

3) It is interesting that, with the terminology of [13], [14], an input/output map over a ring "splits" if and only if its Hankel matrix (and corresponding reachability and observability matrices) admit a WGI. This may have (as yet unexplored) consequences for the realization and identification of split systems (delay-differential, etc.).

4) There is a connection between the results of this paper and theorems of Dolezal and of Silverman and Bucy (see [12]) about matrices over a ring of real-analytic functions in one variable. The results of these authors ensure that an analytic basis exists for the kernel of such a matrix. From this it is possible to deduce the existence of a WGI for a constant-rank matrix over one-variable analytic function rings. In fact, since these rings are "elementary divisor rings" (see [6]) "Smith forms" exist over them, and the results of Bose–Mitra–Batigne mentioned before already apply. Of course, as soon as rings of functions in more than one variable are considered, the methods in this paper become essential.

IV. Generalized Inverses over Certain Rings

The usual (Moore–Penrose) generalized inverse (GI) of a real matrix A is a WGI B which in addition to (1), (2) also satisfies the properties

$$(AB)' = AB \tag{7}$$

$$(BA)' = BA \tag{8}$$

where the prime indicates transpose. For complex matrices this definition is extended with the prime meaning conjugate transpose, but over more arbitrary fields there is in general no satisfactory definition of GI (see for instance [8], [15]). We shall not attempt here to define a GI over arbitrary rings; rather, we shall restrict ourselves to a particular ring, representative of those appearing in various applications.

Consider

$$R = \mathbf{R}[z_1, \cdots, z_n]^*, \tag{9}$$

the ring of rational functions $a(z_1,\cdots,z_n)b(z_1,\cdots,z_n)^{-1}$ with real coefficients and with $b(z_1,\cdots,z_n)\neq 0$ for all $(z_1,\cdots,z_n)$ in $\mathbf{R}^n$. A GI of a matrix A over R is then a matrix B (also over R) satisfying (1), (2), (7), and (8). The standard uniqueness proofs carry over to this case, and the unique such B will be denoted by A^+, if it exists. The main result here is the following.

Theorem 3: The following statements are equivalent for any matrix A over R.

a) *A has a GI.*

b) *A has a WGI.*

c) *A has constant rank over all $(z_1,\cdots,z_n)$ in $\mathbf{R}^n$*

d) *A can be written as PA_0Q, with P,Q unimodular R-matrices (= having determinant $\neq 0$ for all $(z_1,\cdots,z_n)$ in $\mathbf{R}^n$) and*

$$A_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

with I_r being the identity matrix of order $r = \text{rank } A$.

The equivalence of the last three statements is proved in a way analogous to that of Theorem 1, as a consequence of Theorem 2. (In fact, it can be also proved directly from the real version of the former, writing $A = q^{-1}\bar{A}$, with $\bar{A}$ a polynomial matrix and q a polynomial.) To complete the proof of Theorem 3 one then needs only to prove that, say, d) implies a) [since clearly, a) implies b)].

Let P,Q be as in d). Denote

$$C := (I_r, 0)Q$$

$$D := P\begin{pmatrix} I_r \\ 0 \end{pmatrix}.$$

Then $A = DC$, and C,D both have rank r when evaluated at each $(z_1,\cdots,z_n)$ in $\mathbf{R}^n$. Thus the square matrices CC' and $D'D$ are both invertible when evaluated at each $(z_1,\cdots,z_n)$, and hence over the ring R. Defining

$$A^+ := C'(CC')^{-1}(D'D)^{-1}D', \tag{10}$$

it is easy (and standard) to prove that A^+ is the GI of A.

An example of a matrix A for which the above theorem fails is (with $n=1$) $A := (z_1)$. Calculating pointwise the GI of this 1×1 matrix one obtains

$$A(z)^+ = \left(\frac{1}{z}\right) \qquad \text{for } z \neq 0,$$

and

$$A(0)^+ = (0).$$

Since (by uniqueness) $A^+(z) = A(z)^+$, A^+ is not in R (i.e., it is not given by a rational function with a nonzero denominator). The rank of this A degenerates at zero. (Of course, A being here square means that the above machinery is not really needed.)

The implication proved above is valid for other rings of real functions; the basic property needed is that a nowhere zero function be invertible, as for instance with R above or with the ring of real-analytic functions.

Some system-theoretic applications of generalized inverses were mentioned in the paper of Bose with Mitra [4]. The above ring appears naturally when using the theory of systems over rings in the study of "families of linear systems" [7], which arise, for instance, in relation to identification problems or in the study of some large-scale systems [9]. An example of such an application is the following. Assume that

$$x_\lambda(t+1) = F_\lambda x(t+1) + G_\lambda u(t+1) \tag{11}$$

describes the time-evolution of a family of linear systems, one for each value of the vector parameter λ in $\mathbf{R}^n$. Furthermore, assume that the parameterization has a reasonable algebraic structure, in this case, by rational functions of λ with no real poles. Consider the problem of obtaining a minimum-energy control $u(0),\cdots,u(t-1)$ transfering an x_0 into $x(t)=0$ (fixed t). The problem is, for each system (each λ) that of calculating the GI of the reachability matrix

$$\mathbf{R}_\lambda = [G_\lambda, F_\lambda G_\lambda, \cdots, F_\lambda^{t-1}G_\lambda].$$

Rather than calculating the GI for each value λ, it is clearly desirable to be able to calculate $\mathbf{R}_\lambda^+$ *as a function of* λ, having the same algebraic structure, i.e., of computing a GI over the ring R considered above. This is accomplished by the above methods. Other applications are given by the family-of-systems analogs of those in the paper of Lovass-Nagy *et al.* [11].

Remark: There is in fact a simple algorithm to find A^+ when it exists. Let B be a matrix over the *field* of rational functions calculated using formula (10) from a full-rank factorization of the original matrix A over the field. This construction is well-defined since the rank of A is the same locally and over the rational function field; in fact, rational functions over the reals give a field over which the setup in [8] applies. *A prioiri* B is then a matrix whose entries may have real poles. But for every $(z_1,\cdots,z_n)$ not a pole of B, $B(z)$ is the GI of $A(z)$, i.e., $A^+(z)$ (existence of the latter being assured by the theorem). Since such $(z_1,\cdots,z_n)$ form a dense subset of R^n, and since two rational functions equal on a dense set are equal everywhere, then $B = A^+$. Thus, the latter can be calculated in a rather straightforward way.

As a (simple) illustration of the above procedure, let (with $x = z_1, y = z_2$),

$$A = \begin{pmatrix} x & y & x^2+y+1 \\ x^2y & xy^2 & x^3y+xy^2+xy \end{pmatrix}.$$

It is clear that A has rank one at every x,y. Calculating a full rank factorization over the rational function field one obtains

$$P = \begin{pmatrix} 1 & 0 \\ xy & 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} x & y & x^2+y+1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that Q is not unimodular, since its determinant vanishes at $x=0$. However, calculating A^+ through (10) results in

$$A^+ = (ab)^{-1}\begin{bmatrix} 1 & xy \\ y & xy^2 \\ x^2+y+1 & xy(x^2+y+1) \end{bmatrix}$$

where

$$a = CC' = x^2+y^2+(x^2+y+1)^2$$

and

$$b = D'D = 1+(xy)^2.$$

Since both a,b have not real zeros, A^+ is indeed well-defined over the ring R.

V. Conclusions

Weak generalized inverses were studied for matrices (and other linear maps) over integral domains, and a characterization was given of those matrices admitting a WGI. For rings of real functions these definitions can be extended to treat generalized inverses, which possess the usual optimization properties.

As a suggestion for further research, it appears that an interesting next step would be that of studying the existence of other generalizations of A^+, having optimality properties with respect to the rings in question, even if A^+ itself does not exist.

References

[1] D. Baugne, "Note on integral {1}-inverses of integral matrices," Dep. Math., George Washington Univ., preprint, 1977.

[2] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*. New York: Wiley, 1974.

[3] N. K. Bose, Ed., "Multidimensional systems," *Proc. IEEE*, Special Issue, 1977.

[4] N. K. Bose and S. K. Mitra, "Generalized inverse of polynomial matrices," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 491-493, 1978.

[5] N. Bourbaki, *Commutative Algebra*. Reading, MA: Addison-Wesley, 1972.

[6] R. Bumby, E. D. Sontag, H. J. Sussmann, and W. Vasconcelos, "Remarks on the pole-shifting problem over rings," *J. Pure Appl. Algebra*, to be published.

[7] C. Byrnes, "On the control of certain deterministic, infinite-dimensional systems by algebra-geometric techniques," *Amer. J. Math.*, vol. 100, no. 4, 1978.

[8] R. E. Kalman, "Algebraic aspects of the generalized inverse of a rectangular matrix," in *Generalized Inverses and Applications*, A. Nashed, Ed. New York: Academic, 1975.

[9] E. W. Kamen, "Lectures on algebraic system theory: Linear systems over rings," NASA Contractor Rep. 3016, 1978.

[10] D. Quillen, "Projective modules over polynomial rings," *Inv. Math.* vol. 36, pp. 167-171, 1976.

[11] V. Lovass-Nagy, R. J. Miller, and D. L. Powers, "Introduction to the applications of the simplest matrix-generalized inverse in system science," *IEEE Trans. Circuits Syst.*, vol. CAS-25, pp. 776-771, 1978.

[12] L. M. Silverman and R. S. Bucy, "Generalizations of a theorem of Dolezal," *Math. Syst. Theory*, vol. 4, pp. 334-339, 1970.

[13] E. D. Sontag "Linear systems over commutative rings: A survey," *Richerche di Automatica*, vol. 7, pp. 1-34, 1976.

[14] ——, "On split realizations of response maps over rings," *Inform. Contr.*, vol. 37, pp. 23-33, 1978.

[15] J. D. Fulton, "Generalized inverses of matrices over a finite field," *Discrete Math.*, vol. 21, pp. 23-29, 1978.

entry type: 1 CMP 12 : 19
J571 759 (4) 15
Sontag, Eduardo D.
On generalized inverses of
polynomial and other matrices.
IEEE Trans. Automat. Control 25
(1980), no. 3, 514 - 517.

REVIEWER: Donald W. Robinson
(Provo, Utah)
5646

NOV 05 1980

MATHEMATICAL REVIEWS

Reviewer: Please provide 5-character classification(s) according to 19
Subject Index Classification. (See 1978 or later MR Annual Indexes

15A09, _ _ _ _ _

Conventions (use colored pencil):

- Greek : underline in red
- Fraktur : print or type roman letter, underline in green
- Script : print or type roman letter, encircle in blue
- Boldface : underline with a wavy blue line

Do not underline for italics.

PLEASE TYPE WITH SPACE BETWEEN LINES

Necessary and sufficient conditions are given for a matrix over $\mathbb{C}[z_1,\dots,z_n]$ to have a weak generalized inverse. These results are then extended to morphisms of finitely generated projective R-modules over an integral domain R. In particular, such a morphism has a weak generalized inverse if and only if it has the same rank over every R-field.

The author communicated to the reviewer the following addendum, which extends a result of M. H. Pearl [Linear Algebra Appl. 1 (1968) 571-587; MR 40 #158]. With respect to the involution of transpose', a matrix A over an integral domain R has a Moore-Penrose inverse over R if and only if the ranks of A, AA' and A'A are equal and the same over all R-fields.

If in Russian, check if you consider this article of exceptional meri warranting translation. ☐