

On the existence of approximately coprime factorizations for retarded systems

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Abstract: This note establishes a result linking algebraically coprime factorizations of transfer matrices of delay systems to approximately coprime factorizations in the sense of distributions. The latter have been employed by the second author in the study of function-space controllability for such systems.

Keywords: Delay systems; coprime factorizations; controllability; realization.

1. Introduction

This note deals with retarded delay-differential systems (with one commensurable delay) of the type:

$$\frac{d}{dt}x(t) = \sum_{i=0}^N F_i x(t - ih) + \sum_{i=0}^N G_i u(t - ih). \quad (1)$$

There are two ways in which one may ‘algebraize’ this equation. The first is based on the use of the formal delay operator $\sigma x(t) := x(t - h)$ (from here on, h is an arbitrary but fixed positive real number that specifies the unit length of the delay). Here one introduces the matrices

$$F(\sigma) := \sum_{i=0}^N F_i \sigma^i \quad \text{and} \quad G(\sigma) := \sum_{i=0}^N G_i \sigma^i,$$

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and then one views the pair $(F(\sigma), G(\sigma))$ as a system over the ring $\mathbb{R}[\sigma]$ (e.g., [1], [8]). Alternatively, one may introduce the operator $zx(t) := x(t + h)$ and express the ‘transfer matrix’ of (1) as a proper ratio of two matrices

$$Q(s, z) := sz^N I - \sum_{i=0}^N F_i z^{N-i}$$

and

$$P(s, z) := \sum_{i=0}^N G_i z^{N-i}.$$

This second approach is more convenient when studying various issues in realization theory and control. In particular, it is easier then to construct function space models (such as $M_2 = (\mathbb{R} \times L^2[-h, 0])^N$ (e.g., [2])) and to study function space reachability, stabilizability, and other properties [10,11]. Among the latter is the notion of approximate left coprimeness. The pair (Q, P) is said to be *approximately left coprime* if there exist sequences of matrices $\{R_n\}$ and $\{S_n\}$ with entries in a suitable space of Schwartz distributions such that, with the topology of distributions,

$$\check{Q} * R_n + \check{P} * S_n \rightarrow \delta I, \quad (2)$$

where \check{Q} and \check{P} denote the respective inverse Laplace transforms, $*$ denotes convolution, and δ is the delta function (see [10] for details and also [6] for a similar approach).

When this condition is satisfied, the natural observable function space model introduced in [10] turns out to be quasi-reachable (or approximately reachable in a more popular terminology), and it plays a certain crucial role in studying reachability questions of infinite-dimensional systems [11]. The approximate coprimeness condition clearly reduces to the standard coprimeness condition (see, e.g., [3]) for finite-dimensional systems.

Since delay-differential systems are often specified in terms of the first representation, using matrices $F(\sigma)$, $G(\sigma)$, and since such matrices are more directly associated with various algebraic algorithms (e.g., realization algorithms that use Hankel matrices), it is important to relate the two. The relationship between these two approaches is not completely straightforward. As an illustration, take the system

$$\begin{aligned}\dot{x}(t) &= x(t-h) + u(t), \\ y(t) &= x(t-h).\end{aligned}$$

The transfer function of this system is $1/(s\sigma - 1)$, or equivalently $\sigma/(s - \sigma)$ using the operator σ . While the first fractional representation is clearly coprime, the second is not, because there is a common zero $s = \sigma = 0$ between the numerator and denominator.

The main objective of this note is to provide an easy-to-check criterion for the existence of an approximately coprime factorization *stated in terms of the matrices obtained by the ring-theoretic approach*. To achieve this, we first establish some relationships between the usual coprimeness notions in both formalisms.

Notation

$\mathbb{R}[x, y]$ is the ring of polynomials in two variables x, y .

$\mathbb{R}(x, y)$ is the field of rational functions in two variables x, y .

$\mathcal{E}'(\mathbb{R}^-)$ is the convolution algebra consisting of Schwartz distributions having compact support contained in $(-\infty, 0]$ (see [10]). With the interpretation $zx(t) := x(t+h)$, the operator z may be identified with the left-shift convolution operator δ_{-h} (the Dirac delta function placed at point $-h$), and in this way we shall regard $\mathbb{R}[s, z]$ as a subalgebra of $\mathcal{E}'(\mathbb{R}^-)$. Similarly, the operator $\sigma x(t) := x(t-h)$ may be identified with δ_h ; note carefully that with this identification the ring $\mathbb{R}[s, \sigma]$ is *not* a subalgebra of $\mathcal{E}'(\mathbb{R}^-)$, though it is an algebra of distributions.

For every row vector $v(s, \sigma)$ over $\mathbb{R}[s, \sigma]$, the degree

$$\deg_\sigma v(s, \sigma)$$

is the highest degree in σ of its entries. Now let V

be a $p \times n$ matrix over the ring $\mathbb{R}[s, \sigma]$. Write its rows explicitly,

$$V = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix}.$$

The *row degrees* of V (with respect to σ) are the integers

$$\deg_\sigma v_1, \dots, \deg_\sigma v_p.$$

Its *highest order coefficient matrix* $[V]_\sigma$ is defined to be the matrix over $\mathbb{R}[s]$ consisting of the row-wise highest-order coefficients of $\sigma^{\deg_\sigma v_1}, \dots, \sigma^{\deg_\sigma v_p}$. Thus, for example, for

$$V = \begin{pmatrix} s - \sigma & \sigma & 1 \\ s\sigma^2 & s + \sigma & \sigma^2 \end{pmatrix},$$

$$[V]_\sigma = \begin{pmatrix} -1 & 1 & 0 \\ s & 0 & 1 \end{pmatrix}.$$

The matrix V is said to be *row proper* in σ , if $[V]_\sigma$ is of full rank over the field $\mathbb{R}(s)$ of rational functions in s . (For instance, the above example is row proper.)

2. Preliminaries: Coprimeness conditions

Let $W(s, \sigma) := (w_{ij})$ be the $p \times m$ transfer matrix of a retarded delay-differential system with commensurable point delays. By this we mean that W can be written in the form

$$W(s, \sigma) = \frac{\sum_{k=0}^{N_1} H_k(s) \sigma^k}{s^n + \sum_{k=0}^{N_2} g_k(s) \sigma^k} \quad (3)$$

where $g_k(s)$ is a scalar polynomial of degree less than n , and $H_k(s)$ is a matrix of size $p \times m$ with polynomial entries of degree at most $n-1$. Given such a W , we may also view it as a matrix of rational functions in (s, z) , where $z = \sigma^{-1}$.

Thus it is possible to consider two different types of factorizations, one over the ring $\mathbb{R}[s, z]$, and the other over $\mathbb{R}[s, \sigma]$. We now introduce two different notions of coprimeness for such factorizations. As mentioned earlier, the first is related to the function space reachability and the solution of control problems, while the second arises in the ring-theoretic approach to such systems.

Definition 2.1. The pair (Q, P) , $Q \in \mathbb{R}[s, z]^{p \times p}$, $P \in \mathbb{R}[s, z]^{p \times m}$ is said to be an *approximately left coprime factorization* of W if Q is invertible over $\mathbb{R}(s, z)$ and the pair satisfy

- (a) $W = Q^{-1}P$ when W is written as a function of (s, z) ;
- (b) there exists sequences X_n and Y_n of matrices over $\mathcal{E}'(\mathbb{R}^-)$ such that

$$\check{Q} * X_n + \check{P} * Y_n \rightarrow \delta I \quad (4)$$

where \check{Q} and \check{P} denote the inverse Laplace transforms, $*$ denotes convolution, and the convergence is with respect to the topology of $\mathcal{E}'(\mathbb{R}^-)$.

Definition 2.2. The pair (D, N) , $D \in \mathbb{R}[s, \sigma]^{p \times p}$, $N \in \mathbb{R}[s, \sigma]^{p \times m}$ is said to be a *left Bezout factorization* of W if D is invertible over $\mathbb{R}(s, \sigma)$ and the pair satisfy

- (a) $W = D^{-1}N$ when W is written as a function of (s, σ) ;
- (b) there exist matrices X, Y over $\mathbb{R}[s, \sigma]$ such that

$$NX + DY = I. \quad (5)$$

In order to discuss the relationship between the above two notions of coprimeness, we first need to relate a left factorization over $\mathbb{R}[s, \sigma]$ to a left factorization over $\mathbb{R}[s, z]$.

Let $Q^{-1}(s, \sigma)P(s, \sigma)$ be any left factorization of W over $\mathbb{R}[s, \sigma]$, i.e.,

$$W = Q^{-1}(s, \sigma)P(s, \sigma),$$

$$Q \in \mathbb{R}[s, \sigma]^{p \times p}, \quad P \in \mathbb{R}[s, \sigma]^{p \times m}.$$

Substituting $\sigma = z^{-1}$ and clearing denominators, we obtain of course a factorization over $\mathbb{R}[s, z]$. More precisely, we proceed as follows: Let r_1, \dots, r_p be the row degrees of the composite matrix

$$\begin{bmatrix} Q(s, \sigma) & P(s, \sigma) \end{bmatrix}$$

with respect to the variable σ . Then the pair of matrices

$$\begin{bmatrix} \check{Q} & \check{P} \end{bmatrix} := \begin{pmatrix} z^{r_1} & & & \\ & \ddots & & \\ & & & z^{r_p} \end{pmatrix} \cdot \begin{bmatrix} Q(s, z^{-1}) & P(s, z^{-1}) \end{bmatrix} \quad (6)$$

give a left factorization over the ring $\mathbb{R}[s, z]$. We

call this factorization *the left factorization associated to the pair (Q, P)* .

As noted in the Introduction, if there exists an approximately left coprime factorization of W , then the canonical realization can be constructed with an M_2 -like function state space [10,11]; to be more precise, the Fuhrmann-type *standard observable* realization Σ^Q (the notation is as in [10]) turns out to be quasi-reachable. *Then, how can we check the existence of such a factorization?* This will be the theme of the next section.

3. Existence of an approximately left coprime factorization

For the existence of a left Bezout factorization over $\mathbb{R}[s, \sigma]$, the following criterion in terms of the Hankel matrix is known:

Theorem 3.1 [9,4,5]. *Let W be a transfer matrix given by (3). Expand W in negative powers of s as follows:*

$$W = \sum_{k=1}^{\infty} W_k(\sigma) s^{-k} \quad (7)$$

where each $p \times m$ matrix W_k is a polynomial matrix in σ . With $\{W_k\}$, form the Hankel matrix

$$\mathcal{H} := \begin{pmatrix} W_1 & W_2 & W_3 & \cdots \\ W_2 & W_3 & W_4 & \cdots \\ W_3 & W_4 & W_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let k be the rank of \mathcal{H} over the field $\mathbb{R}(\sigma)$ of rational functions in σ . Then W admits a left Bezout factorization if and only if all $k \times k$ minors of \mathcal{H} have no common zero $\sigma \in \mathbb{C}$.

See [7] for related material. Note also that the condition above is equivalent to saying that the rank of \mathcal{H} is independent of σ .

Though we may obtain a left Bezout factorization of W from this result, the associated pair need not be approximately left coprime. The reason is that there can be a ‘cancellation at $\sigma = \infty$ ’ between the obtained matrices $Q(s, \sigma)$ and $P(s, \sigma)$.

To avoid this, we need the following lemma:

Lemma 3.2. *The pair $(Q(s, z), P(s, z))$, $Q, P \in \mathbb{R}[s, z]$, is approximately left coprime if (and only if)*

- (i) $\text{rank}[Q(s, e^{hs}) \ P(s, e^{hs})] = p$ for all $s \in \mathbb{C}$;
and
- (ii) $\text{rank}[Q(s, 0) \ P(s, 0)] = p$ for some $s \in \mathbb{C}$.

Proof. Let \check{Q}, \check{P} be the inverse Laplace transforms of Q and P , respectively. According to Corollary 4.10 of [11], (Q, P) is approximately left coprime if and only if condition (i) holds and $[\check{Q} \ \check{P}]$ generates a full rank matrix at the origin (considered over the field of fractions of $\mathcal{E}'(\mathbb{R}^-)$). Since each entry of $[Q \ P]$ is of the form

$$\sum_{i=0}^n \phi_i(s) z^i, \quad z = e^{hs},$$

the origin $\{0\}$ is an isolated point of the support of $[\check{Q} \ \check{P}]$, and it generates an atomic distribution $[Q_0 \ P_0]$ at $\{0\}$. Clearly $[Q_0 \ P_0]$ is precisely the inverse Laplace transform of the coefficient matrix of z^0 in $[Q(s, z) \ P(s, z)]$, which is

$$[Q(s, 0) \ P(s, 0)].$$

Thus $[Q_0 \ P_0]$ generates a full rank distribution if and only if $[Q(s, 0) \ P(s, 0)]$ has a minor which is not identically zero, and hence the lemma is proved. \square

We are now ready to give the following theorem:

Theorem 3.3. *Assume that the the Hankel matrix \mathcal{H} satisfies the conditions of Theorem 3.1. Let $W = Q^{-1}P$ be a Bezout factorization over $\mathbb{R}[s, \sigma]$, whose existence is guaranteed by Theorem 3.1. Suppose that the pair (Q, P) is row proper with respect to σ . Then the pair $(\check{Q}(s, z), \check{P}(s, z))$, associated with (Q, P) gives an approximately left coprime factorization of W .*

Proof. In view of $\sigma = z^{-1}$ and by the way $(\check{Q}(s, z), \check{P}(s, z))$ is defined by (6), the coefficient matrix of z^0 of $(\check{Q}(s, z), \check{P}(s, z))$ is given by the highest order coefficient matrix of $[Q(s, \sigma) \ P(s, \sigma)]$, i.e.,

$$[Q(s, \sigma) \ P(s, \sigma)]_{\sigma}.$$

Therefore, by Lemma 3.2 above, the pair $(\check{Q}(s, z), \check{P}(s, z))$ is approximately left coprime. \square

Remark 3.4. By [10], [11], condition (i) in Lemma 3.2 means that the standard observable realization Σ^Q (see Section 2) associated to the factorization $Q^{-1}P$ is spectrally reachable, i.e., every element in any generalized eigenspace is reachable. Also, $\text{rank } Q(s, 0) = \text{full}$ is equivalent to the state space to be eigenfunction complete [11, Corollary 3.13], and condition (ii) in Lemma above then means that the system can be made eigenfunction complete by suitable feedback (which does not change reachability) [11, Theorem 4.1]. It is interesting to observe that we have obtained:

1. this standard realization (associated with $Q^{-1}P$) is eigenfunction complete if and only if $Q(s, \sigma)$ is row proper;
2. it can be made eigenfunction complete by feedback if and only if (Q, P) is row proper.

If (Q, P) is not row proper, there is still a possibility to convert the pair to a new one where the theorem may apply:

Lemma 3.5. *Suppose the pair $(Q(s, \sigma), P(s, \sigma))$ is not row proper, but*

$$[Q(s, \sigma) \ P(s, \sigma)]_{\sigma}$$

is a constant matrix. Then $(Q(s, \sigma), P(s, \sigma))$ can be reduced to a new Bezout factorization $(Q_1(s, \sigma), P_1(s, \sigma))$ which has less row degrees in σ .

Proof. Precisely in the same way as in the finite-dimensional case (e.g., [3]), we can reduce the row degrees of $[Q(s, \sigma) \ P(s, \sigma)]$ by premultiplying a suitable unimodular matrix in σ . Clearly this procedure does not change the Bezout property. \square

Thus, we may apply Theorem 3.3 to the new pair (Q_1, P_1) . Since many retarded systems are specified in the form

$$Q(s, \sigma) = sI - F(\sigma),$$

$$P(s, \sigma) = G(\sigma),$$

this condition is often satisfied, although there is a retarded delay-differential system whose $[Q(s, \sigma) \ P(s, \sigma)]_{\sigma}$ depends on s . See Example 4.2.

Another important difference between the present situation and the finite-dimensional case is that the reduced pair (Q_1, P_1) need not be brought to a row proper one by repeating the procedure above, because the highest order coefficient matrix of the new factorization may well involve terms depending on s . See Example 4.3 for a more detailed discussion.

There is another way in which the above results can be understood. This is as follows.

Definition 3.6. The pair (D, N) , $D \in \mathbb{R}[s, \sigma]^{p \times p}$, $N \in \mathbb{R}[s, \sigma]^{p \times m}$ is said to be a *strong left Bezout factorization* of W if it is a left Bezout factorization and in addition it holds that (Q, P) is row proper with respect to σ .

In informal terms, the above says that a strong factorization is one which is Bezout even at infinity. Then Theorem 3.3 says simply: *If W admits a strong left Bezout factorization then it also admits an approximately left coprime factorization.*

We do not as yet have an elegant characterization of when strong Bezout factorizations exist. But since any two Bezout factorizations differ at most by multiplication by a unimodular matrix, the following is a partial result in that direction.

Theorem 3.7. *Assume that the the Hankel matrix \mathcal{H} satisfies the conditions of Theorem 3.1. Let (Q, P) be any Bezout factorization, whose existence is guaranteed by the Theorem. Then, W admits a strong left Bezout factorization if and only if there exists a $p \times p$ unimodular matrix $A(s, \sigma)$ such that*

$$A(s, \sigma)[Q \ P] \quad (8)$$

is row proper.

4. Examples

Example 4.1. Consider the impulse response matrix

$$W = \frac{1}{s^2 - s\sigma - \sigma} \begin{pmatrix} s - \sigma \\ 1 \end{pmatrix} \quad (9)$$

which admits the power series expansion

$$W = \begin{pmatrix} 1 \\ 0 \end{pmatrix} s^{-1} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} s^{-2} + \begin{pmatrix} \sigma \\ \sigma \end{pmatrix} s^{-3} + \dots \quad (10)$$

Its Hankel matrix \mathcal{H} is

$$\mathcal{H} = \begin{pmatrix} 1 & 0 & \sigma & \dots \\ 0 & 1 & \sigma & \dots \\ 0 & \sigma & \sigma^2 & \dots \\ 1 & \sigma & \sigma + \sigma^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (11)$$

so that the 2 by 2 minors of \mathcal{H} have no common zeros. Hence there exists a Bezout factorization. Indeed, the pair

$$Q = \begin{pmatrix} s & -\sigma \\ -1 & s - \sigma \end{pmatrix}, \quad P = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (12)$$

gives a Bezout factorization. However, the pair is not row proper because

$$[Q \ P]_{\sigma} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad (13)$$

and hence its associated pair is *not* approximately left coprime.

But the condition of Lemma 3.5 is satisfied, and we can obtain a row proper pair by multiplying

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad (14)$$

from the left. This leads to an approximately coprime pair

$$Q_1 = \begin{pmatrix} s+1 & -s \\ -z & sz-1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (15)$$

The following example gives a retarded system whose highest order coefficient matrix depends on s .

Example 4.2. Consider

$$W(s, \sigma) = \frac{1}{s(s + \sigma)}. \quad (16)$$

In this example, the delay-differential equation representing $W(s, \sigma)$ is

$$\ddot{x}(t) + \dot{x}(t-h) - u(t) = 0.$$

This appears to be of neutral type, but it is indeed represented by

$$\dot{x}_1(t) = x_2(t),$$

$$\dot{x}_2(t) = -x_2(t-h) + u(t),$$

which is retarded type. The matrix $[s^2 + s\sigma, 1]_{\sigma}$ is $[s, 0]$, so that it is not a constant matrix.

Example 4.3. Now consider the pair

$$Q = \begin{pmatrix} s - \sigma^2 & \sigma^2 \\ s^2 + \sigma^3 + 1 & (1 - \sigma)s - \sigma^3 + 1 \end{pmatrix}, \quad (17a)$$

$$P = \begin{pmatrix} 1 \\ -\sigma \end{pmatrix}. \quad (17b)$$

Then $[Q \ P]_\sigma$ is

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad (18)$$

so that the pair (Q, P) is not row proper. We can reduce the row degree (three) of the second row by premultiplying

$$\begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix}. \quad (19)$$

This gives

$$\begin{pmatrix} s - \sigma^2 & \sigma^2 & 1 \\ s^2 + s\sigma + 1 & -s\sigma + s + 1 & 0 \end{pmatrix}, \quad (20)$$

whose highest order coefficient matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ s & -s & 0 \end{pmatrix}. \quad (21)$$

depends on s , even though (18) does not, and we cannot reduce the row degrees any further in this example, because the row degree of the first row is one higher than that of the second row.

Although the system is not of retarded type, there seems to be no characterization of retarded systems known at present, which can be used to exclude such a situation.

5. Concluding remarks

Some relationships between coprimeness conditions in the distribution approach and the ring approach have been established. In short, coprimeness is preserved along with row-properness, but it is not necessarily so if the latter condition is not satisfied. One can regard row-properness as a coprimeness condition at infinity; *if we consider a stronger notion of coprimeness on the Riemann sphere, i.e., no common zeros including infinity, then coprimeness in this sense is always preserved.*

We did not give a result on the converse of Theorem 3.3. This is because coprimeness in the (s, z) -domain is in the *approximate* sense (note that $[Q, P]_\sigma$ may vanish for some s). Of course, if

a left factorization $Q(s, z)^{-1}P(s, z)$ satisfies a Bezout identity over $\mathbb{R}[s, z]$, and if the highest order coefficient matrix $[Q(s, z) \ P(s, z)]_z$ has full rank for every s , then an analogous result as Theorem 3.3 holds, since factorization over $\mathbb{R}[s, z]$ and $\mathbb{R}[s, \sigma]$ are related essentially by $z \leftrightarrow \sigma^{-1}$. However, in this case, no concise characterization on the existence of a Bezout factorization over $\mathbb{R}[s, z]$ as Theorem 3.1 seems to be known at present.

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